Section 6.0: Introduction

Binary programming is a form of integer programming. The word "binary" refers to the decision variables. When the decision variables are binary, this means that they can only take on the values of either 0 or 1. That might seem overly restrictive, but there are many situations that can easily be modeled using binary decision variables. For example, the following decisions could be modeled with binary decision variables:

- Should we locate a new automobile dealership at this location?
- Should I choose to apply to this college?
- Should I invest in this stock?

Q1. What are the possible answers to each of these questions?

Q2. How would could 0 and 1 be used to model those answers?

To explore the idea of binary decision variables further, a brief introductory example is given. In this example, Jarvis is choosing whether to work on two projects.

Section 6.0.1: Jarvis Selects Projects

Jarvis is considering working on two short-term projects, one with a $5,000 profit and the other with a $7,000 profit. Project 1 requires 7 days, and project 2 requires 11 days.

To formulate this problem, Jarvis needs to define some decision variables. A binary decision variable is an integer decision variable that has been further restricted to just two values, 0 and 1. Jarvis will use two **binary decision variables**, one for each project. A value of 1 means he decides to work on the project; a value of 0 means he decides not to.

Jarvis develops the following problem formulation:

Decision Variables

Let: x_1 = the binary decision variable for Project 1
x_2 = the binary decision variable for Project 2
z = the amount of profit

Objective Function

Maximize: $z = 5{,}000x_1 + 7{,}000x_2$

Constraints

Subject to:

Time Available: $7x_1 + 11x_2 \leq A$, where A is the number of days Jarvis is available to work on the projects

Binary Decision Variables: x_1 equals 0 or 1; x_2 equals 0 or 1

Q3. What are the four possible feasible solutions for this problem?

Q4. What is the value of the objective function for each of these possible solutions?

In order to determine the optimal solution, Jarvis needs to know how many days he has available to work on the projects. He considers several possible values for A. Suppose A = 5 days. In this case, there is not enough time to complete either project, so the optimal solution is $x_1 = 0$ and $x_2 = 0$.

Q5. For what other values of A is $x_1 = 0$ and $x_2 = 0$ the optimal solution?

Q6. What is the smallest value of A that changes the optimal solution?
  a. What is the new optimal solution?
  b. What other values of A generate the same optimal solution?

Q7. What is the smallest value of A that causes the optimal solution to change again?
  a. What is the optimal solution for that value of A?
  b. What other values of A produce that same optimal solution?

Q8. What is the smallest value of A that changes the optimal solution a third time?
  a. What is that new optimal solution?
  b. What other values of A produce that same optimal solution?

Q9. Explain why continuing to increase A cannot change the optimal solution again.

Therefore, sometimes increasing the value of A does not the optimal solution. Other times, increasing the value of A does change the optimal solution. In this case, the constraint is non-binding.

For example, consider A = 10 days. That is, Jarvis has 10 days available to work on the projects. In a *linear programming* example, there would be no value in increasing this resource above 10 days (assuming it stays within the allowable increase for the constraint). Furthermore, reducing this resource by up to 3 days would also not impact the optimal solution. As a result, a $0 shadow price would be reported because this constraint is non-binding.

In *linear programming*, the decision variables are continuous. Thus, if there's slack, getting more of the resources does not increase the value.

In *binary programming*, if there's slack, getting more resources may or may not increase the optimal solution. That is, if Jarvis has 11 days—rather than 10 days—available, the optimal solution changes. However, if he has 10 days—rather than 9 days—available, the optimal solution does not change.

In other words, if A were 10, the optimal solution clearly would be to do Project 1, as there is not enough time to do Project 2. The constraint would not be binding because it will have three days of slack. However, there would be value in increasing the number of days available from 10 days to 11 days. If that were done, Jarvis could now complete Project 2 and earn $7,000 instead of only $5,000, and the constraint would then be binding. For this reason, Solver does not generate Sensitivity Reports for binary programming problems, like integer programming problems.

Since one extra day could potentially earn him $7,000 rather than $5,000, Jarvis may try to reschedule his time to provide an extra day to work on the project.

Q10. What is the marginal value of the extra day of work?

Section 6.1: Flipping Houses

Dream Homes, Inc. is a company that buys houses that are in need of repair. The houses are renovated, updated, and then sold for a profit. This process is referred to as *house flipping*.

Mr. Dale, a retired realtor, owns Dream Homes, Inc. and wants to provide a summer job for each of his five grandchildren, all of whom are in college. Mr. Dale would pay each grandchild by dividing the profits.

Each of Mr. Dale's five grandchildren—Ani, Benita, Cameron, Dante, and Edwin—has specific skills appropriate for house restoration. The skills for each grandchild are shown in Table 6.1.1. Notice that all the grandchildren are capable of cleaning.

Grandchild	**Skills**		
Ani	Plumbing	Carpentry	Cleaning
Benita	Plumbing	Landscaping	Cleaning
Cameron	Painting	Carpentry	Cleaning
Dante	Painting	Landscaping	Cleaning
Edwin	Landscaping	Carpentry	Cleaning

Table 6.1.1: List of each grandchild and their skills

Mr. Dale has identified ten available houses on the market. Each house has an estimated profit margin. This estimate is based on the difference between the total cost and the estimated resale value of the house. The total cost includes the purchase price, the cost of necessary materials, and the expense of hiring any outside contractor assistance. Then, this total cost is subtracted from the estimated resale value of the house to find the estimated total profit. Mr. Dale's goal is to earn the highest possible total profit.

The grandchildren are all college students, so they have just 12 weeks in which to work. They are focused on the goal of acquiring the most lucrative houses so they can earn the most profit. The grandchildren first need to select which houses to flip in order to yield the highest profit. They have all agreed to work 8 hours a day, 6 days a week.

Each house requires the appropriate skill set of the grandchildren (i.e., plumbing, carpentry, landscaping, painting, and cleaning). Mr. Dale needs to determine which house, or set of houses, will allow the grandchildren to earn the largest profit over the course of the summer.

6.1.1: Problem Formulation

Mr. Dale decides the maximum amount that he can invest in this summer venture for his grandchildren's benefit is $500,000. This includes the costs to purchase the homes and the materials, and to pay any contractors' fees. Since he plans on investing in more than one house, the question becomes: in which of the ten available houses should Mr. Dale invest to maximize the profit for his grandchildren?

The formulation of this problem utilizes *binary decision variables*. A binary decision variable is defined for each house under consideration:

- A value of 1 indicates that a particular house will be purchased
- A value of 0 indicates that particular house will not be purchased.

Mr. Dale lets the decision variable x_1 represent whether or not house 1 is purchased; he lets the decision variable x_2 represent whether or not house 2 is purchased, and so on. For example, if the value of x_1 is 1, then Mr. Dale will purchase house 1; if the value of x_1 is 0, then Mr. Dale will not purchase house 1.

Furthermore, Mr. Dale lets the variable P_1 represent the estimated profit of house 1; he lets the variable P_2 represent the estimated of house 2, and so on. Thus, the objective function is:

$$z = P_1 \cdot x_1 + P_2 \cdot x_2 + P_3 \cdot x_3 + P_4 \cdot x_4 + P_5 \cdot x_5 + P_6 \cdot x_6 + P_7 \cdot x_7 + P_8 \cdot x_8 + P_9 \cdot x_9 + P_{10} \cdot x_{10}.$$

Q1. Explain what the value of z represents in this problem context. How do you know?

A more compact way to write this objective function uses summation notation: $z = \sum_{i=1}^{10} P_i \cdot x_i$. See Appendix A for an explanation of summation notation.

Mr. Dale seeks to maximize z subject to constraints related to total cost and available labor.

Q2. What constraints must Mr. Dale consider?

Mr. Dale is constrained by the total cost. The total cost restriction takes into consideration the cost to purchase the home, the costs for necessary material, and the costs to any outside contractors. This total cannot exceed $500,000.

Mr. Dale lets:

H_i = the cost to purchase house i,
M_i = the cost of materials for house i, and
C_i = any contractors' costs for house i.

Then, the *estimated* total cost for a particular house, i, is $H_i + M_i + C_i$.

The actual total cost for that particular house depends on whether they decide to purchase it. The binary decision variable x_i equals 1 if the house is purchased and 0 if not. Therefore, multiplying the potential total cost by x_i gives that actual total cost for a particular house: $(H_i + M_i + C_i) \cdot x_i$. .

For example, if Mr. Dale purchase house 2, the total cost for this house is $(H_2 + M_2 + C_2) \cdot 1 = H_2 + M_2 + C_2$. On the other hand, if Mr. Dale does not purchase house 2, the total cost is: $(H_2 + M_2 + C_2) \cdot 0 = 0$ That is, it will not cost Mr. Dale anything if he does not purchase the house.

Finally, using summation notation, the total cost constraint is:

$$\sum_{i=1}^{10} \left(H_i + M_i + C_i \right) \cdot x_i \le 500{,}000.$$

Q3. In your own words, explain the meaning of this constraint.

Next, Mr. Dale writes the constraints on the number of hours of labor available for each type of work (i.e., plumbing, carpentry, landscaping, painting, and cleaning). To do so, he must analyze the availability and skills of each grandchild. Each grandchild has committed to working a maximum of 8 hours a day, 6 days a week, for 12 weeks during summer vacation. This amounts to each grandchild being able to work a maximum of 576 hours.

Table 6.1.1 shows the skills for each grandchild. Since there are only two grandchildren skilled at plumbing (Ani and Benita) and each is available for a maximum of 576 hours, Mr. Dale must ensure that the required plumbing hours are less than or equal to 2 · 576 = 1,152 hours.

Similarly, the total painting hours cannot exceed 1,152 hours because only two grandchildren are skilled at painting (Cameron and Dante).

There are three grandchildren who are skilled at landscaping as well as carpentry, so he can assign a maximum of 3 · 576 = 1,728 hours to each of those trades.

Finally, all the grandchildren are able to clean. The hours for plumbing, painting, landscaping, and carpentry have been accounted for. Therefore, Mr. Dale only needs to ensure that the total number of hours does not exceed 5 · 576 = 2,880, the number of hours his grandchildren are available to work.

Given all of these factors, Mr. Dale lets:

p_i = the number of plumbing hours required for house i,
q_i = the number of painting hours required for house i,
r_i = the number of landscaping hours required for house i,
s_i = the number of carpentry hours required for house i,
t_i = the total number of hours required for house i.

Then, he has the following labor constraints:

$$\sum_{i=1}^{10} p_i \cdot x_i \leq 1{,}152 \text{ hours} \text{ (plumbing labor constraint)},$$

$$\sum_{i=1}^{10} q_i \cdot x_i \leq 1{,}152 \text{ hours} \text{ (painting labor constraint)},$$

$$\sum_{i=1}^{10} r_i \cdot x_i \leq 1{,}728 \text{ hours} \text{ (landscaping labor constraint), and}$$

$$\sum_{i=1}^{10} s_i \cdot x_i \leq 1{,}728 \text{ hours} \text{ (carpentry labor constraint)}.$$

There is also a constraint on the total amount of time available:

$$\sum_{i=1}^{10} t_i \cdot x_i \leq 2{,}880 \text{ hours} \text{ (total time available)}.$$

Notice that the total cleaning hours is not really a constraint by itself, because it is included in the constraint that restricts the total hours worked. Each of the five grandchildren can work a maximum of 576 hours, and thus the total amount of labor spread over all areas cannot exceed 5 · 576 = 2,880 hours.

Q4. In your own words, explain why there is no need for a constraint on cleaning hours.

In order to complete this formulation, Mr. Dale needs to add in the information about the houses he may purchase. First, Table 6.1.2 shows the information about cost and total potential value for each house.

	House 1	House 2	House 3	House 4	House 5	House 6	House 7	House 8	House 9	House 10
Potential Value	$125,000	$135,000	$178,000	$110,000	$108,000	$124,000	$244,000	$192,000	$130,000	$275,000
Cost (H_i)	$65,000	$100,000	$125,000	$70,000	$35,000	$99,000	$140,000	$115,000	$88,000	$129,000
Materials Cost (M_i)	$5,000	$3,000	$3,750	$5,500	$7,500	$2,000	$8,000	$6,000	$7,000	$16,000
Contractors Cost (C_i)	$6,000	$3,000	$8,000	$1,000	$17,500	$0	$10,000	$9,300	$0	$22,000

Table 6.1.2: Value and cost of each house

Q5. Calculate the net profit for each house by adding up all the costs and subtracting this value from the potential value of the house.

Q6. Based on this information, predict which houses Mr. Dale will purchase. Explain your reasoning.

Next, Mr. Dale collects the information about the labor hours for each of the houses. This is shown in Table 6.1.3.

	House 1	House 2	House 3	House 4	House 5	House 6	House 7	House 8	House 9	House 10
Plumbing Hours	238	211	264	145	211	100	400	422	185	304
Painting Hours	150	125	115	80	130	50	250	160	100	200
Landscaping Hours	210	175	161	112	182	70	350	224	140	280
Carpentry Hours	264	330	143	242	165	180	220	385	330	396
Cleaning Hours	300	280	310	410	335	200	350	325	390	325

Table 6.1.3: Labor hours for each house

Q7. Write the complete problem formulation.

6.1.2: Solving the Problem

Next, Mr. Dale creates a spreadsheet with the problem formulation from the previous section. The spreadsheet is set up in a way similar to the previous chapters. However, Mr. Dale needs to add an additional constraint to tell Solver that the decision variables are binary. To do so, he chooses "bin" from the drop-down menu, as shown in Figure 6.1.1.

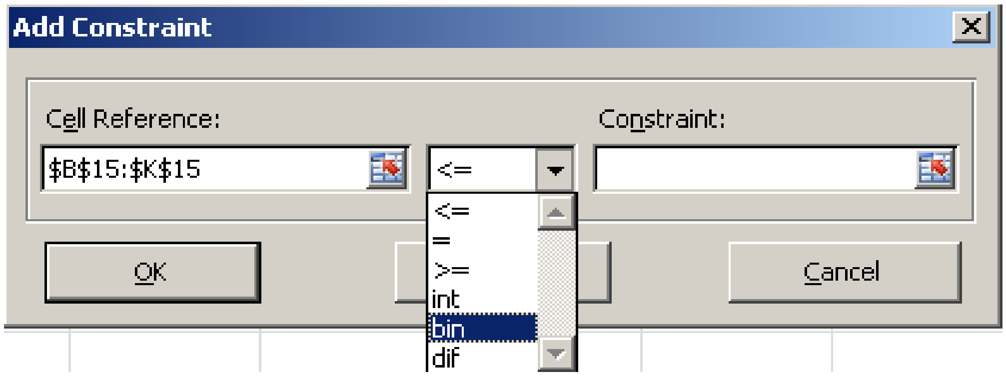


Figure 6.1.1: Specifying binary constraints

Mr. Dale's solution is shown in Figure 6.1.2. Use this spreadsheet to answer the following questions.

Q8. What is the largest profit the grandchildren can make during the summer flipping houses?

Q9. Which houses will be purchased to produce this profit?

Q10. How much money will each person receive if the profit is divided equally among the grandchildren?

The optimal solution identifies two houses to be purchased and flipped to maximize profits. One of these houses has the largest margin for profit. But the second home in the optimal solution is fourth out of the possible ten homes when ranked based on profit margin.

Q11. Why do you suspect this has happened?

In binary programming, decision variables change in units of one. That means they consume resources in discrete increments. To explore this phenomenon, Mr. Dale changes the values of two decision variables, without resolving the problem. He changes the binary decision value for house 1 from a 1 to a 0. Then he changes the binary decision value for house 7—the second most profitable house—from a 0 to a 1. These changes are shown in Figure 6.1.3.

Q12. Looking at the spreadsheet in Figure 6.1.3, explain why Mr. Dale cannot purchase house 10 and house 7.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	Chapter 6: Binary Programming													
2	6.2 House Flipping													
3	Maximization Profit													
4														
5		House 1	House 2	House 3	House 4	House 5	House 6	House 7	House 8	House 9	House 10			
6	Total Value ($)	$125,000	$135,000	$178,000	$110,000	$108,000	$124,000	$244,000	$192,000	$130,000	$275,000			
7	Cost ($)	$65,000	$100,000	$125,000	$70,000	$35,000	$99,000	$140,000	$115,000	$88,000	$129,000			
8	Materials Cost ($)	$5,000	$3,000	$3,750	$5,500	$7,500	$2,000	$8,000	$6,000	$7,000	$16,000			
9	Contractors Cost ($)	$6,000	$3,000	$8,000	$1,000	$17,500	$0	$10,000	$9,300	$0	$22,000			
10	Total Cost ($)	$76,000	$106,000	$136,750	$76,500	$60,000	$101,000	$158,000	$130,300	$95,000	$167,000	$243,000	≤	$500,000
11	**Net Profit ($)**	**$49,000**	**$29,000**	**$41,250**	**$33,500**	**$48,000**	**$23,000**	**$86,000**	**$61,700**	**$35,000**	**$108,000**			
12														
13														
14	Decision Variable	House 1 (x_1)	House 2 (x_2)	House 3 (x_3)	House 4 (x_4)	House 5 (x_5)	House 6 (x_6)	House 7 (x_7)	House 8 (x_8)	House 9 (x_9)	House 10 (x_{10})			
15	Decision Variable Values	1	0	0	0	0	0	0	0	0	1			
16														**Total Profit ($)**
17	Objective Function [Max. Profit ($)]	$49,000	$29,000	$41,250	$33,500	$48,000	$23,000	$86,000	$61,700	$35,000	$108,000			$157,000.00
18														
19	**Constraints**													
20	Maximum Plumbing Hours (hours)	238	211	264	145	211	100	400	422	185	304	542	≤	1152
21	Maximum Painting Hours (hours)	150	125	115	80	130	50	250	160	100	200	350	≤	1152
22	Maximum Landscaping Hours (hours)	210	175	161	112	182	70	350	224	140	280	490	≤	1728
23	Maximum Carpentry Hours (hours)	264	330	143	242	165	180	220	385	330	396	660	≤	1728
24	Maximum Cleaning Hours (hours)	300	280	310	410	335	200	350	325	390	325			
25	**Total Labor Hours**	**1162**	**1121**	**993**	**989**	**1023**	**600**	**1570**	**1516**	**1145**	**1505**	2667	≤	2880

Figure 6.1.2: A flipping houses spreadsheet with the optimal solution

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	Chapter 6: Binary Programming													
2	6.2 House Flipping													
3	Maximization Profit													
4														
5		House 1	House 2	House 3	House 4	House 5	House 6	House 7	House 8	House 9	House 10			
6	Total Value ($)	$125,000	$135,000	$178,000	$110,000	$108,000	$124,000	$244,000	$192,000	$130,000	$275,000			
7	Cost ($)	$65,000	$100,000	$125,000	$70,000	$35,000	$99,000	$140,000	$115,000	$88,000	$129,000			
8	Materials Cost ($)	$5,000	$3,000	$3,750	$5,500	$7,500	$2,000	$8,000	$6,000	$7,000	$16,000			
9	Contractors Cost ($)	$6,000	$3,000	$8,000	$1,000	$17,500	$0	$10,000	$9,300	$0	$22,000			
10	Total Cost ($)	$76,000	$106,000	$136,750	$76,500	$60,000	$101,000	$158,000	$130,300	$95,000	$167,000	$325,000	≤	$500,000
11	Net Profit ($)	$49,000	$29,000	$41,250	$33,500	$48,000	$23,000	$86,000	$61,700	$35,000	$108,000			
12														
13														
14	Decision Variable	House 1 (x_1)	House 2 (x_2)	House 3 (x_3)	House 4 (x_4)	House 5 (x_5)	House 6 (x_6)	House 7 (x_7)	House 8 (x_8)	House 9 (x_9)	House 10 (x_{10})			
15	Decision Variable Values	0	0	0	0	0	0	1	0	0	1			
16														Total Profit ($)
17	Objective Function [Max. Profit ($)]	$49,000	$29,000	$41,250	$33,500	$48,000	$23,000	$86,000	$61,700	$35,000	$108,000			$194,000.00
18														
19	Constraints													
20	Maximum Plumbing Hours (hours)	238	211	264	145	211	100	400	422	185	304	704	≤	1152
21	Maximum Painting Hours (hours)	150	125	115	80	130	50	250	160	100	200	450	≤	1152
22	Maximum Landscaping Hours (hours)	210	175	161	112	182	70	350	224	140	280	630	≤	1728
23	Maximum Carpentry Hours (hours)	264	330	143	242	165	180	220	385	330	396	616	≤	1728
24	Maximum Cleaning Hours (hours)	300	280	310	410	335	200	350	325	390	325			
25	Total Labor Hours	1162	1121	993	989	1023	600	1570	1516	1145	1505	3075	≤	2880

Figure 6.1.3: A flipping houses spreadsheet with Houses 10 and 7 chosen

As a result, Mr. Dale has identified the fact that the group's ability to make a profit is restricted by the total available hours, even though the constraint is not binding. He explores this restriction to see whether it is advantageous to hire a cleaning crew.

To explore this issue, Mr. Dale increases the available hours in 100-hour increments until he sees a change in the optimal solution. The first time a change occurs is when the total labor hours is 3080 (a 200-hour increase). The new optimal solution is shown in Figure 6.1.4.

Q13. What is the new largest profit the grandchildren can make during the summer flipping houses?

Q14. How much money will each person receive if the profit is divided equally among the grandchildren?

Q15. Which houses will be purchased to produce this profit?

Q16. Is this new optimal solution what you expected? Why or why not?

Q17. Would you recommend that Mr. Dale hires a cleaning crew so that he has an extra 200 labor hours available?

Q18. Suppose Mr. Dale decides to hire a cleaning crew. How much per hour should he pay the cleaning crew? Explain your reasoning.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	Chapter 6: Binary Programming													
2	6.2 House Flipping													
3	Maximization Profit													
4														
5		House 1	House 2	House 3	House 4	House 5	House 6	House 7	House 8	House 9	House 10			
6	Total Value ($)	$125,000	$135,000	$178,000	$110,000	$108,000	$124,000	$244,000	$192,000	$130,000	$275,000			
7	Cost ($)	$65,000	$100,000	$125,000	$70,000	$35,000	$99,000	$140,000	$115,000	$88,000	$129,000			
8	Materials Cost ($)	$5,000	$3,000	$3,750	$5,500	$7,500	$2,000	$8,000	$6,000	$7,000	$16,000			
9	Contractors Cost ($)	$6,000	$3,000	$8,000	$1,000	$17,500	$0	$10,000	$9,300	$0	$22,000			
10	Total Cost ($)	$76,000	$106,000	$136,750	$76,500	$60,000	$101,000	$158,000	$130,300	$95,000	$167,000	$325,000	≤	$500,000
11	**Net Profit ($)**	**$49,000**	**$29,000**	**$41,250**	**$33,500**	**$48,000**	**$23,000**	**$86,000**	**$61,700**	**$35,000**	**$108,000**			
12														
13														
14	Decision Variable	House 1 (x_1)	House 2 (x_2)	House 3 (x_3)	House 4 (x_4)	House 5 (x_5)	House 6 (x_6)	House 7 (x_7)	House 8 (x_8)	House 9 (x_9)	House 10 (x_{10})			
15	Decision Variable Values	0	0	0	0	0	0	1	0	0	1			
16														**Total Profit ($)**
17	Objective Function [Max. Profit ($)]	$49,000	$29,000	$41,250	$33,500	$48,000	$23,000	$86,000	$61,700	$35,000	$108,000			$194,000.00
18														
19	**Constraints**													
20	Maximum Plumbing Hours (hours)	238	211	264	145	211	100	400	422	185	304	704	≤	1152
21	Maximum Painting Hours (hours)	150	125	115	80	130	50	250	160	100	200	450	≤	1152
22	Maximum Landscaping Hours (hours)	210	175	161	112	182	70	350	224	140	280	630	≤	1728
23	Maximum Carpentry Hours (hours)	264	330	143	242	165	180	220	385	330	396	616	≤	1728
24	Maximum Cleaning Hours (hours)	300	280	310	410	335	200	350	325	390	325			
25	**Total Labor Hours**	**1162**	**1121**	**993**	**989**	**1023**	**600**	**1570**	**1516**	**1145**	**1505**	3075	≤	3080

Figure 6.1.4: New optimal solution when total labor hours is increased by 200 hours

Section 6.2: Sam Johnson Makes a Hard Decision

Sam Johnson lives in Brooklyn, New York. He is deciding which colleges to apply to this fall. He has taken the SAT, and his total score is 1650. He wants to find colleges that are more than 50 miles from home but no more than 300 miles from home. He also wants to find colleges with acceptance rates of at least 50%. Sam has lived in a large city all of his life and enjoys life in the city, so he would like to find a college in or near an urban setting. He also prefers a medium-sized school.

Sam decides to do some Internet searches to learn more about potential colleges that satisfy his criteria. Along with the web sites for individual schools, he finds these resources useful:

- http://collegesearch.collegeboard.com/search/index.jsp
- http://www.uscollegesearch.org
- http://www.collegeview.com/collegesearch/index.jsp
- http://cnsearch.collegenet.com/cgi-bin/CN/index

Sam decides that the most important criteria to him are distance from home, tuition, average debt at graduation, acceptance rate, and size. Table 6.2.1 contains Sam's short list of 11 schools that fit his criteria. His problem is that his mom and dad will pay the application fee for only five colleges. Sam needs to decide where he should apply.

School	Location	Distance from Home	Tuition	Average Debt at Graduation	Acceptance Rate	Size
Western Connecticut State University	Danbury, CT	55 miles	$15,344	No data available	57%	6,001 students
Rutgers: Camden Regional Campus	Camden, NJ	80 miles	$18,263	$18,645	53%	49,760 students
Drexel University	Philadelphia, PA	97 miles	$28,780	No data available	76%	20,821 students
College of Saint Rose	Albany, NY	141 miles	$20,620	$24,732	68%	5,000 students
Johnson & Wales University	Providence, RI	155 miles	$21,717	$19,890	81%	16,095 students
Worcester State College	Worcester, MA	157 miles	$11,619	$13,742	56%	5,470 students
Loyola College in Maryland	Baltimore, MD	169 miles	$34,250	$16,073	64%	6,131 students
University of Massachusetts Boston	Boston, MA	190 miles	$19,977	$14,805	63%	13,433 students
Suffolk University	Boston, MA	190 miles	$24,250	No data available	69%	5,196 students
University of Massachusetts Lowell	Lowell, MA	194 miles	$19,714	$14,833	70%	11,635 students
Syracuse University	Syracuse, NY	197 miles	$31,686	$24,000	51%	19,082 students

Table 6.2.1: Sam's list of 11 colleges meeting his criteria with their data

6.2.1: Developing the Constraints

As seen in Table 6.2.1, Sam collected data on the criteria most important to him: distance from home, tuition, average debt at graduation, acceptance rate, and size.. However, he is not able to learn the average debt at graduation for several of the colleges on his list, so he decides not to use that criterion in his selection process.

Now, Sam wants to decide which colleges to apply to. He bases this decision on the colleges' acceptance rates and on attractiveness scores that he will determine. The attractiveness scores will be based on each college's distance from his home, its tuition, and some personal preferences.

To calculate the attractiveness score for each college, Sam relies on some techniques used in the Multi-Criteria Decision Making process from Chapter 1. Specifically, he performs the following steps:

1. Rescale scores by assigning a score between zero and one for each criterion.
2. Weight scores by multiplying the criterion scores by their assigned weight.
3. Add these three weighted scores to obtain the attractiveness score for each school.

First, Sam rescales the scores for each criterion. Table 6.2.2 shows how he will assign the scores for distance, tuition, and size.

Distance (weight = 0.2)		Tuition (weight = 0.2)		Size (weight = 0.2)	
Range (miles)	Score	Range ($)	Score	Range (students)	Score
50 – 90	0	< 10,000	1	< 5,000	0
91 – 130	0.33	10,000 – 14,999	0.67	5,000 – 9,999	0.8
131 – 170	0.67	15,000 – 19,999	0.33	10,000 – 19,999	1
≥ 171	1	≥ 20,000	0	≥ 20,000	0.2

Table 6.2.2: Sam's scheme for assigning scores

Sam also creates personal preference scores based on what he learned from his Internet searches about the quality of life on each campus. These scores are shown in Table 6.2.3.

School	**Personal Preference Scores**
Western Connecticut State University	0.14
Rutgers: Camden Regional Campus	0.71
Drexel University	0.61
College of Saint Rose	0.63
Johnson & Wales University	0.33
Worcester State College	0.44
Loyola College in Maryland	0.4
University of Massachusetts Boston	0.81
Suffolk University	0.6
University of Massachusetts Lowell	0.72
Syracuse University	0.53

Table 6.2.3: Sam's personal preference scores for each college

Second, Sam wants to give equal weight to the objective criteria (distance from home, tuition, and size) and a higher weight to his personal preferences. He decides to weight distance, tuition, and size at 20% each and his personal preferences at 40%.

Third, Sam computes the attractiveness score for each school as follows:

(0.2)(distance score) + (0.2)(tuition score) + (0.2)(size score) + (0.4)(personal preference score).

Sam will determine which colleges to apply to based on these attractiveness scores as well as on the acceptance rates of the colleges. Sam wants to be fairly certain that he gets accepted into at least two of the schools to which he is applying. To ensure this happens, he introduces a constraint that will force at least two of the schools in his portfolio will have acceptance rates of at least 70%. Furthermore, Sam is seeking balance in acceptance rates. He wants to apply to schools where he will be challenged, so he creates a constraint requiring the average of all the acceptance rates at the schools where he will apply to be at least 60%.

In a similar way, Sam could balance his preferences for distance from home, size, and cost of the colleges. He notes that he could set up additional constraints to ensure that he is applying to a wide range of schools when considering any criterion, but he decides not to do so at this time.

Below are the constraints on Sam's portfolio of colleges that he decides to use:

- At least *two* colleges with acceptance rates of at least 70% should be selected.
- The average acceptance rate of all the selected colleges should be at least 60%.
- He will apply to exactly *five* colleges.

6.2.2: Formulating the Problem

To formulate this problem, Sam must define some variables. The first variables he defines are **binary decision variables.** He uses binary decision variables to determine which schools to apply to. As shown in the previous sections, a binary decision variable is an integer decision variable that has been further restricted to just two values: 0 or 1. Sam will use 11 binary decision variables, one for each school he is considering. Sam's problem formulation will maximize his objective function, subject to constraints. When the problem is solved, the binary decision variables for those colleges that will appear in Sam's portfolio (i.e., the schools Sam *will* apply to) will be assigned a value of one. The binary decision variables for those colleges that will not appear in Sam's portfolio (i.e., the schools Sam *will not* apply to) will be assigned a value of zero.

Therefore, Sam defines the following decision variables:

Let: x_1 = the binary decision value for college 1 (Western Connecticut State University)
x_2 = the binary decision value for college 2 (Rutgers: Camden Regional Campus)
x_3 = the binary decision value for college 3 (Drexel University)
x_4 = the binary decision value for college 4 (College of Saint Rose)
x_5 = the binary decision value for college 5 (Johnson & Wales University)
x_6 = the binary decision value for college 6 (Worcester State College)
x_7 = the binary decision value for college 7 (Loyola College in Maryland)
x_8 = the binary decision value for college 8 (University of Massachusetts Boston)
x_9 = the binary decision value for college 9 (Suffolk University)
x_{10} = the binary decision value for college 10 (University of Massachusetts Lowell)
x_{11} = the binary decision value for college 11 (Syracuse University)

To organize his data, Sam also defines some variables for acceptance rate, size, tuition, distance from home, and attractiveness. Table 6.2.4 lists these variables.

Variables	Description
R_i	Acceptance rate (%) for school *i*
S_i	Size (# of students) of school *i*
T_i	Tuition ($) for school *i*
H_i	Distance (miles) from home to school *i*
A_i	Attractiveness score for school *i*

Table 6.2.4: The variables Sam will use

Finally, Sam defines a **binary indicator coefficient**. Like binary decision variables, a binary indicator coefficient is restricted to two values: 0 or 1. To develop a binary indicator coefficient, Sam thinks about checking each of the 11 colleges in his list against the list of the three constraints appearing at the end of the previous section. For each college, its data must be checked to see if it meets each of the constraints.

For example, the first constraint is that at least two of the colleges selected have acceptance rates greater than or equal to 70%. So, for each college, he must ask the question, "Is the acceptance rate greater than or equal to 70%?" This is a yes-or-no question, and this is where binary indicator coefficients come into play. If the answer is yes, the binary indicator will be assigned a value of 1, but if the answer is no, the binary indicator will be assigned a value of 0.

Sam defines the following binary indicator coefficient:
Let: $_bR_\mathrm{i}$ = the binary indicator for R_i

These binary indicators will be used to count the number of schools in the portfolio that satisfy the constraint. In Sam's formulation, $_bR_i$ represents whether school *i* satisfies the first constraint (i.e., whether its acceptance rate is greater than or equal to 70%). Referring to Table 6.2.1, the first school on the list is Western Connecticut State University, and its acceptance rate (R_1) is 57%. So it does not meet the first constraint. Therefore, the binary indicator is assigned the value zero: $_bR_1 = 0$. Similarly, $_bR_2 = 0$ because Rutgers's acceptance rate is also less than 70%. However, the third school in the list, Drexel University, has an acceptance rate of 76%. Therefore, $_bR_3 = 1$. The values of $_bR_i$ are shown in Table 6.2.5.

School	Acceptance Rate	Binary Indicator
Western Connecticut State University	57%	$_bR_1 = 0$
Rutgers: Camden Regional Campus	53%	$_bR_2 = 0$
Drexel University	76%	$_bR_3 = 1$
College of Saint Rose	68%	$_bR_4 = 0$
Johnson & Wales University	81%	$_bR_5 = 1$
Worcester State College	56%	$_bR_6 = 0$
Loyola College in Maryland	64%	$_bR_7 = 0$
University of Massachusetts Boston	63%	$_bR_8 = 0$
Suffolk University	69%	$_bR_9 = 0$
University of Massachusetts Lowell	70%	$_bR_{10} = 1$
Syracuse University	51%	$_bR_{11} = 0$

To complete the formulation, Sam defines an objective function that takes into account the acceptance rates (R_i) and attractiveness scores (A_i)—which are based on tuition, distance from home, size, and personal preferences—for the selected schools.

Maximize: $z = (R_1 \cdot A_1)x_1 + (R_2 \cdot A_2)x_2 + (R_3 \cdot A_3)x_3 + (R_4 \cdot A_4)x_4 + (R_5 \cdot A_5)x_5 + (R_6 \cdot A_6)x_6$
$+ (R_7 \cdot A_7)x_7 + (R_8 \cdot A_8)x_8 + (R_9 \cdot A_9)x_9 + (R_{10} \cdot A_{10})x_{10} + (R_{11} \cdot A_{11})x_{11}$

As in the previous section, this sum can be written using summation notation:

$$z = \sum_{i=1}^{11} \left(R_i \cdot A_i \right) x_i \ .$$

This objective function sums 11 products, one for each school. Each product consists of three factors: the binary decision variable x_i representing whether a particular school is selected, that school's acceptance rate R_i, and its attractiveness score A_i. The summation of these products represents the total value of Sam's portfolio. Since exactly five schools will be selected, six of the values of x_i will be 0. Thus, those resulting products will also be 0. The sum of the products $R_i \cdot A_i$ must be maximized to determine which schools will be selected.

Finally, Sam's three constraints will be:

At least *two* schools with acceptance rates greater than or equal to 70% should be selected:

$$\sum_{i=1}^{11} {}_bR_i \cdot x_i \geq 2$$

The average acceptance rate of all the selected colleges should be at least 60%:

$$\frac{\sum_{i=1}^{11} R_i \cdot x_i}{5} \geq 0.6$$

Sam will apply to exactly five colleges:

$$\sum_{i=1}^{11} x_i = 5$$

In the first constraint, the binary indicator coefficient (${}_bR_i$) for each school is multiplied by the binary decision variable x_i for that school. Since both factors in this constraint are binary, every one of the products also is binary. If the particular school does not meet the constraint, then the left hand side of the constraint is 0. If the school does meet the constraint, then it will be 1. Therefore, the sum of those 11 products is the number of schools in the portfolio that meet the particular constraint (e.g., acceptance rate greater than or equal to 70%). Notice that the binary indicator ${}_bR_i$ is crucial to the formulation of this constraint.

In the second constraint, the average acceptance rate is the sum of the acceptance rates at the selected colleges $\left(\sum_{i=1}^{11} R_i \cdot x_i \right)$ divided by the number of colleges selected for the portfolio (5).

In the last constraint, the sum of all of the binary decision variables x_i is just the number of schools in the portfolio. This value must be 5 because Sam will apply to exactly 5 schools.

Now, Sam has the complete problem formulation:

Decision Variables

Let:

x_1 = the binary decision value for college 1 (Western Connecticut State University)
x_2 = the binary decision value for college 2 (Rutgers: Camden Regional Campus)
x_3 = the binary decision value for college 3 (Drexel University)
x_4 = the binary decision value for college 4 (College of Saint Rose)
x_5 = the binary decision value for college 5 (Johnson & Wales University)
x_6 = the binary decision value for college 6 (Worcester State College)
x_7 = the binary decision value for college 7 (Loyola College in Maryland)
x_8 = the binary decision value for college 8 (University of Massachusetts Boston)
x_9 = the binary decision value for college 9 (Suffolk University)
x_{10} = the binary decision value for college 10 (University of Massachusetts Lowell)
x_{11} = the binary decision value for college 11 (Syracuse University)
z = the total value of the portfolio

Objective Function

Maximize: $$z = \sum_{i=1}^{11} \left(R_i \cdot A_i \right) x_i$$

Subject to:

Constraints

At least *two* schools with acceptance rates greater than or equal to 70% should be selected:

$$\sum_{i=1}^{11} {}_bR_i \cdot x_i \geq 2$$

The average acceptance rate of all the selected colleges should be at least 60%:

$$\frac{\sum_{i=1}^{11} R_i \cdot x_i}{5} \geq 0.6$$

Sam will apply to exactly five colleges:

$$\sum_{i=1}^{11} x_i = 5$$

6.2.3: Solving the Problem

The process of checking data against constraints and assigning values to the binary variables would be very tedious and time-consuming if done by hand. Fortunately spreadsheet solvers allow the user to define variables as binary, just as they allow the definition of integer variables. Figure 6.2.1 shows Sam's spreadsheet formulation after he has set up and solved the problem.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	Chapter 6: Binary Programming													
2	6.2 Sam Johnson Makes a Hard Decision													
3	Maximize Total Value of Portfolio													
4														
5	Variable	H_i	T_i	S_i					A_i	R_i	bR_i			
6	School	Distance from home (miles)	Tuition ($)	Size (# of students)	Tuition score (20%)	Distance score (20%)	Size score (20%)	Personal preference (40%)	Total attractiveness score	Acceptance rate (%)	Acceptance rate indicator (1 if ≥70%, 0 otherwise)	Objective function coefficient (acceptance rate * total attractiveness score)	Decision variable value	Objective function value
7	Western Connectic	58	15,344	6,001	0.33	0	0.8	0.14	0.28	0.57	0	0.161	0	0.00
8	Rutgers Camden	78	18,263	49,760	0.33	0	0.2	0.61	0.35	0.53	0	0.186	0	0.00
9	Drexel U	97	28,780	20,821	0	0.33	0.2	0.71	0.39	0.76	1	0.296	0	0.00
10	College of Saint Ro	141	20,620	5,000	0	0.67	0.8	0.63	0.55	0.68	0	0.371	1	0.37
11	Johnson & Wales	155	21,717	16,095	0	0.67	1	0.33	0.47	0.81	1	0.377	1	0.38
12	Worcester State Co	157	11,619	5,470	0.67	0.67	0.8	0.44	0.60	0.56	0	0.338	0	0.00
13	Loyola U in MD	169	34,250	6,131	0	0.67	0.8	0.4	0.45	0.64	0	0.291	0	0.00
14	U Mass Boston	189	19,977	13,433	0.33	1	1	0.81	0.79	0.63	0	0.498	1	0.50
15	Suffolk University	191	24,250	5,196	0	1	0.8	0.72	0.65	0.69	0	0.447	1	0.45
16	U Mass Lowell	194	19,714	11,635	0.33	1	1	0.53	0.68	0.7	1	0.475	1	0.47
17	Syracuse U	197	31,686	19,082	0	1	1	0.6	0.64	0.51	0	0.326	0	0.00
18														
19														
21								LHS	Test	RHS				
22		Number of schools						5	=	5		Objective Function		
23		Number whose acceptance rate is >70%						2	≥	2		2.17		
24		Average acceptance rate						0.702	≥	0.6				

Figure 6.2.1: Sam's spreadsheet showing the optimal portfolio of schools to which to apply

Use Sam's Solver spreadsheet to answer the following questions.

Q1. A coefficient for the objective function was computed for each college (see Column L in Figure 6.2.1). Which columns in the spreadsheet were used to do this?

Q2. Why do most of the values in the column labeled "Objective function value" equal 0.00?

Q3. Which schools are included in Sam's portfolio?

Q4. Did the five selected schools have the largest objective function coefficients? Why/why not?

Q5. Which of the selected schools met the acceptance rate constraint?

Q6. What is the average (mean) acceptance rate of the five selected colleges?

Q7. What is the range of the distance from home of the five selected colleges?

Q8. What is the median tuition of the five selected colleges?

Q9. What is the average (mean) size of the five selected colleges?

Section 6.3: An Application of Binary Programming—Assignment Problems

Assignment problems are a special case of binary programming. An assignment problem arises whenever one type of thing, such as a person, must be assigned to another type of thing, such as a task. To solve an assignment problem, a **cost matrix** is usually defined based on the parameters of the given problem. The following is the general form of a cost matrix:

$$C = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & c_{2,3} & \cdots & c_{2,n} \\ c_{3,1} & c_{3,2} & c_{3,3} & \cdots & c_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{m,1} & c_{m,2} & c_{m,3} & \cdots & c_{m,n} \end{bmatrix}$$

Suppose m workers must be assigned to n tasks. Then, each entry of the cost matrix, $c_{i,j}$ could represent the time needed by worker i to perform task j.

For example, suppose five workers need to be assigned to four tasks. The goal is to finish the tasks as quickly as possible. Then, the following cost matrix can be used, where the entries represent the time, in minutes, it takes each worker to complete each task.

$C =$	shaping	assembling	painting	inspecting
worker 1	10	8	14	17
worker 2	13	9	17	16
worker 3	14	11	20	18
worker 4	9	10	15	14
worker 5	14	12	16	16

Q1. Based on this cost matrix, which workers do you think should perform which tasks? Explain your reasoning.

To explore assignment problems further, two problem contexts are discussed in this section. In the first context, a coach of a girls' swimming team must decide which swimmer to assign to each part of a 400-yard medley relay. In the second context, a teacher wants to determine how to assign her students into groups for a class project. In the first context, a cost matrix is utilized, but in the second, a special set of matrices is developed. Then, binary programming is used to find the optimal solution.

6.3.1 Coach Bass's Problem

Bill Bass, coach of the Jefferson High School girls' swimming team, has four swimmers he always assigns to compete in the 400-yard medley relay. In this event, there are four 100-yard legs that each must be swum by a different competitor using a different stroke. The four legs in the relay are butterfly, backstroke, breaststroke, and freestyle.

Coach Bass knows the best times for each of his swimmers for each leg of the relay. He wonders what way of assigning the four swimmers to the four legs of the relay would be the best, based on their best times.

Q2. Other than using best times, what other criteria could Coach Bass use to assign the swimmers?

Q3. Formulate a general cost matrix for Coach Bass's problem, with entries represented by $c_{i,j}$.

Q4. In the case of the medley relay team, what would $c_{1,2}$ represent? What would $c_{i,j}$ represent?

Q5. What is Coach Bass's objective in the medley relay problem?

Q6. What could possibly complicate this problem?

Formulating the Problem

Coach Bass must assign each of the four swimmers to one of the four 100-yard legs in the 400-yard medley relay: butterfly, backstroke, breaststroke, and freestyle. Coach Bass has decided to use the best times for each of his swimmers for each of the legs of the relay to determine which swimmer to use for which leg of the relay. He wants to assign the four swimmers so as to complete the four legs of the relay in the minimum total time. He wonders what assignment would be the best, based on their best times. Table 6.3.1 contains those best times for the four swimmers.

		Leg			
		100-yd butterfly	100-yd backstroke	100-yd breaststroke	100-yd freestyle
Swimmer	Schmidt	59.59	59.83	72.83	52.61
	Reid	60.45	59.56	74.14	53.31
	Sanchez	61.84	64.63	73.69	53.70
	Lamartina	62.37	59.13	74.36	54.77

Table 6.3.1: Best times (in seconds) for four swimmers in the medley relay legs

Q7. Coach Boss first considered using the best swimmer in each event. Why does this strategy not work?

To solve this problem, Coach Bass decides to use Excel Solver. First, he needs a formulation of the problem. To do that, he will use a binary decision variable to represent the possibility that each of the four swimmers could be assigned to any one of the four legs. He defines the following binary decision variables:

Let: $x_{1,1}$ = the binary decision value for swimmer 1 (Schmidt) in leg 1 (butterfly)
$x_{1,2}$ = the binary decision value for swimmer 1 (Schmidt) in leg 2 (backstroke)
$x_{1,3}$ = the binary decision value for swimmer 1 (Schmidt) in leg 3 (breaststroke)
$x_{1,4}$ = the binary decision value for swimmer 1 (Schmidt) in leg 4 (freestyle)
$x_{2,1}$ = the binary decision value for swimmer 2 (Reid) in leg 1 (butterfly)
$x_{2,2}$ = the binary decision value for swimmer 2 (Reid) in leg 2 (backstroke)
$x_{2,3}$ = the binary decision value for swimmer 2 (Reid) in leg 3 (breaststroke)
$x_{2,4}$ = the binary decision value for swimmer 2 (Reid) in leg 4 (freestyle)

Q8. Write the remaining binary decision variables.

Writing out these binary decision values is very long and repetitive. Therefore, Coach Bass decides to use a shortcut, using subscripts:

For $1 \le i \le 4$ and $1 \le j \le 4$, let $x_{i,j}$ = the binary decision value representing whether swimmer i is assigned to leg j.

Q9. How many binary decision variables are there in this problem formulation?

Next, Coach Bass needs to define an objective function in terms of the decision variables. His objective is to minimize the time for the relay team to swim the event. Thus, for each of the four swimmers, he will need her best time in each of the four legs. So, for $1 \le i \le 4$ and $1 \le j \le 4$, let $t_{i,j}$ = the best time for swimmer i in leg j (note that the values of $t_{i,j}$ can be found in Table 6.3.1).

Q10. How are all of the $t_{i,j}$ values different from all of the $x_{i,j}$ values?

Next, using $x_{i,j}$ and $t_{i,j}$ Coach Bass will represent the total time for the medley relay event. For each possible pair of values for i and j, he will multiply $x_{i,j}$ by $t_{i,j}$. Most of the time, $x_{i,j}$ will equal 0 and the corresponding product will also be 0. There are 16 binary decision variables, but only four of them are going to equal 1. These represent the four swimmers who are assigned to swim the four legs of the relay. In those cases, the product $x_{i,j} \cdot t_{i,j}$ will be the best time of the swimmer for the leg she is assigned to swim. Finally, adding all of the products, $x_{i,j} \cdot t_{i,j}$, gives the total of those four best times. Thus, the objective function that Coach Bass would like to minimize is:

$$z = \sum_{i=1}^{4} \left(\sum_{j=1}^{4} x_{i,j} \cdot t_{i,j} \right).$$

What does this complicated-looking double-summation mean? Looking first at the summation outside the parentheses, it is changing the values of i. In other words, it is changing the rows in a matrix such as this:

$$\begin{bmatrix} x_{1,1} \cdot t_{1,1} & x_{1,2} \cdot t_{1,2} & x_{1,3} \cdot t_{1,3} & x_{1,4} \cdot t_{1,4} \\ x_{2,1} \cdot t_{2,1} & x_{2,2} \cdot t_{2,2} & x_{2,3} \cdot t_{2,3} & x_{2,4} \cdot t_{2,4} \\ x_{3,1} \cdot t_{3,1} & x_{3,2} \cdot t_{3,2} & x_{3,3} \cdot t_{3,3} & x_{3,4} \cdot t_{3,4} \\ x_{4,1} \cdot t_{4,1} & x_{4,2} \cdot t_{4,2} & x_{4,3} \cdot t_{4,3} & x_{4,4} \cdot t_{4,4} \end{bmatrix}$$

So, when $i = 1$, the summation inside the parentheses is computed. Looking at the inside summation, it changes the values of j, which are the columns. So, row-by-row, this double summation computes the sum of the products for each row and then adds those row sums to get the sum of all 16 of those products.

The sum can be written out as follows:

$$z = \sum_{i=1}^{4} \left(\sum_{j=1}^{4} x_{i,j} \cdot t_{i,j} \right) = x_{1,1} \cdot t_{1,1} + x_{1,2} \cdot t_{1,2} + x_{1,3} \cdot t_{1,3} + x_{1,4} \cdot t_{1,4} + x_{2,1} \cdot t_{2,1} + \ldots + x_{4,3} \cdot t_{4,3} + x_{4,4} \cdot t_{4,4}$$

But that's the sum of a bunch of zeros plus the best times of the four swimmers who are actually assigned to swim the legs of the medley relay. In other words, that double-summation computes exactly the quantity that Coach Bass wants to minimize!

Q11. Arbitrarily assign each swimmer to a leg of the race.

a. Calculate the value of z for this assignment.

b. Interpret the meaning of z in terms of the problem context.

Finally, Coach Bass needs to add the constraints to the formulation. The constraints apply to the assignment of swimmers and can be stated simply as:

- Every leg of the relay must have exactly one swimmer assigned to it.
- Every swimmer must be assigned to exactly one leg of the relay.

Because each leg of the relay must have exactly one swimmer assigned to it, there are four different leg constraints. Since the legs of the relay are represented by the subscript *j*, the four leg constraints can be written as follows:

$$x_{1,1} + x_{2,1} + x_{3,1} + x_{4,1} = 1$$
$$x_{1,2} + x_{2,2} + x_{3,2} + x_{4,2} = 1$$
$$x_{1,3} + x_{2,3} + x_{3,3} + x_{4,3} = 1$$
$$x_{1,4} + x_{2,4} + x_{3,4} + x_{4,4} = 1$$

Notice that in each of the four leg constraints, the second subscript, representing the leg, has a fixed value. Then the first subscript, representing the swimmers, takes on each value from 1 to 4. Finally, the value of the second subscript is changed and the process is repeated until the second subscript has also taken on each value from 1 to 4. In a more compact notation, these four constraints can be written as follows:

For j = 1, 2, 3, and 4, $\sum_{i=1}^{4} x_{i,j} = 1$.

Similarly, because every swimmer must be assigned to exactly one event, there are also four different swimmer constraints. Since the four swimmers are represented by the subscript *i*, the 4 swimmer constraints can be written:

$$x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = 1$$
$$x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} = 1$$
$$x_{3,1} + x_{3,2} + x_{3,3} + x_{3,4} = 1$$
$$x_{4,1} + x_{4,2} + x_{4,3} + x_{4,4} = 1$$

Once again, a more compact notation is:

For i = 1, 2, 3, and 4, $\sum_{j=1}^{4} x_{i,j} = 1$.

Putting the various parts of the formulation all together, Coach Bass has the following.

<u>Decision Variables</u>

Let

$x_{i,j}$ = the binary decision value representing whether swimmer i is assigned to leg j, for $1 \le i \le 4$ and $1 \le j \le 4$

z = the total of the best times of the four swimmers assigned to swim the relay

<u>Objective Function</u>

Minimize: $z = \sum_{i=1}^{4}\left(\sum_{j=1}^{4} x_{i,j} \cdot t_{i,j}\right)$, where $t_{i,j}$ = the best time for swimmer i in leg j.

Subject to:

<u>Constraints</u>

$$\text{for } j = 1, 2, 3, \text{ and } 4, \ \sum_{i=1}^{4} x_{i,j} = 1$$

$$\text{for } i = 1, 2, 3, \text{ and } 4, \ \sum_{j=1}^{4} x_{i,j} = 1$$

Putting the Formulation into a Spreadsheet

Coach Bass creates a spreadsheet formulation of the medley relay team problem, as shown in Figure 6.3.1. In this spreadsheet, he creates two matrices. One matrix has the best times in each leg for each swimmer ($t_{i,j}$). The other matrix—called the *assignment matrix*—contains the decision variables ($x_{i,j}$). The assignment of swimmers to the legs of the relay will be recorded in this assignment matrix.

At the beginning of the solution process, no swimmer has yet been assigned to any leg of the relay. Thus, each cell in this assignment matrix appears empty (i.e., the value of each of the decision variables it contains is 0).

	A	B	C	D	E	F	G	H	I
1	Chapter 6: Binary Programming								
2	6.3.1 Coach Bass's Problem								
3	Relay Time Minimization								
4									
5	**Swimmers' Times ($t_{i,j}$)**								
6			Butterfly (sec)	Backstroke (sec)	Breaststroke (sec)	Freestyle (sec)			
7		Schmidt	59.59	59.83	72.83	52.61			
8		Reid	60.45	59.56	74.14	53.31			
9		Sanchez	61.84	64.63	73.69	53.70			
10		Lamartina	62.37	59.13	74.36	54.77		**Total time (sec)**	
11								0	
12	**Assignments ($x_{i,j}$)**								
13			Butterfly	Backstroke	Breaststroke	Freestyle			
14		Schmidt					0	=	1
15		Reid					0	=	1
16		Sanchez					0	=	1
17		Lamartina					0	=	1
18			0	0	0	0			
19			=	=	=	=			
20			1	1	1	1			

Figure 6.3.1: Spreadsheet formulation of the Medley Relay Problem

The numbers to the right of the assignment matrix are the row sums. They represent the four swimmer constraints. Since each swimmer must be assigned to swim exactly one leg, each swimmer's row of decision variables must contain one 1 and three 0s. Therefore, in a valid assignment, each row sum must equal 1.

Similarly, the numbers below the assignment matrix are the column sums. They represent the four leg constraints. Each leg of the relay must have exactly one swimmer assigned to it. As a result, in a valid assignment, each column will contain one 1 and three 0s. Thus, each column sum must also equal 1.

To tell Excel to add up values, the SUM function is used. For example, in cell G14, Coach Bass writes the formula:

=SUM(C14:F14).

This formula tells Excel to add up the values in cells C14, D14, E14, and F14.

The only other constraint is that each of the decision variables must be binary. This constraint is added in the Solver window, so Coach Bass does not need to write this information into the spreadsheet at this time.

The objective function is the sum of the best times of the four swimmers who are assigned to the medley relay. It is computed in cell H11 by multiplying each cell in the assignment matrix by the corresponding cell of the swimmers' times matrix and adding all of the products. For example, the cell containing Schmidt's best time in the Butterfly (59.59) is multiplied by the cell in the assignment matrix opposite Schmidt's name and below "Butterfly". In Figure 6.3.1, this means that the value in cell C7 is multiplied by the value in cell C14. Then, each of these products is added together.

This can easily be done using the SUMPRODUCT function of Excel. That is, in cell H11, Coach Bass types

=SUMPRODUCT(C7:F10,C14:F17).

This function will allow him to find the total of the minimum times of the four swimmers assigned to the medley relay.

The current value of the objective function is 0 because no swimmer has yet been assigned to swim any leg of the medley relay. At this point, Coach Bass does not have a valid assignment. None of the rows or columns sum to 1. He must use Solver to find the optimal assignment.

Solving the Problem

Coach Bass sets up Solver as in previous chapters. He makes sure to include the binary constraints on the decision variables.

Figure 6.3.2 shows the spreadsheet after the Solver has found a solution.

	A	B	C	D	E	F	G	H	I
1	Chapter 6: Binary Programming								
2	6.3.1 Coach Bass's Problem								
3	Relay Time Minimization								
4									
5	**Swimmers' Times ($t_{i,j}$)**								
6			Butterfly (sec)	Backstroke (sec)	Breaststroke (sec)	Freestyle (sec)			
7		Schmidt	59.59	59.83	72.83	52.61			
8		Reid	60.45	59.56	74.14	53.31			
9		Sanchez	61.84	64.63	73.69	53.70			
10		Lamartina	62.37	59.13	74.36	54.77		**Total time (sec)**	
11								245.72	
12	**Assignments ($x_{i,j}$)**								
13			Butterfly	Backstroke	Breaststroke	Freestyle			
14		Schmidt	1	0	0	0	1	=	1
15		Reid	0	0	0	1	1	=	1
16		Sanchez	0	0	1	0	1	=	1
17		Lamartina	0	1	0	0	1	=	1
18			1	1	1	1			
19			=	=	=	=			
20			1	1	1	1			

Figure 6.3.2: Spreadsheet showing a solution to Coach Bass's Problem

Q12. What is the optimal assignment of swimmers to legs of the medley relay?

Q13. In this optimal assignment, which swimmers were the fastest for the swim strokes they were assigned? Which swimmers were not the fastest for the strokes assigned to them? Explain why this can occur in the optimal solution.

Q14. What is the minimum total of the best times for each of the swimmers in the legs to which they have been assigned?

A Complication: More Swimmers

In the previous section, Coach Bass had to solve a *balanced assignment problem*. He had the same number of swimmers to assign as he had legs of the relay. What if the problem were out of balance? For example, suppose that Coach Bass has a large and very competitive swim team. Suppose there are eight girls in competition for the four legs of the medley relay. Table 6.3.2 contains the best times for each of the eight girls in each of the four events.

		Leg			
		100-yd butterfly	100-yd butterfly	100-yd butterfly	100-yd butterfly
Swimmer	Schmidt	59.59	59.83	72.83	52.61
	Reid	60.45	59.56	74.14	53.31
	Sanchez	61.84	64.63	73.69	53.70
	Lamartina	62.37	59.13	74.36	54.77
	Wu	60.33	64.30	72.74	54.05
	Greene	62.41	59.03	72.19	56.61
	Kleinfeld	62.43	67.63	74.05	55.55
	Lepinski	59.44	65.06	72.74	52.49

Table 6.3.2: Best times (in seconds) for eight swimmers in the medley relay legs

Much as he did in the first problem, Coach Bass sets up a spreadsheet solver to tackle the problem. First, he needs a problem formulation. He will use a binary decision variable to represent whether each of the eight swimmers is assigned to any one of the four legs.

So, for $1 \le i \le 8$ and $1 \le j \le 4$, let $x_{i,j}$ = the binary decision variable representing whether swimmer i is assigned to leg j.

Q15. How many binary decision variables are there in this problem formulation?

Next, Coach Bass defines an objective function in terms of the decision variables. His objective is still to minimize the time for his relay team to swim the event. Thus, for each of the eight swimmers, he will need her best time in each of the four legs. So, he lets $t_{i,j}$ = the best time for swimmer i in leg j, for $1 \le i \le 8$ and $1 \le j \le 4$.

Next, using $x_{i,j}$ and $t_{i,j}$, Coach Bass represents the total time for the medley relay event. For each pair of values for i and j, he multiplies $x_{i,j}$ by $t_{i,j}$.. Finally, adding all of the products, $x_{i,j} \cdot t_{i,j}$, $x_{i,j} \cdot t_{i,j}$, gives him the total of those four best times. Therefore, the objective function is:

$$z = \sum_{i=1}^{8} \left(\sum_{j=1}^{4} x_{i,j} \cdot t_{i,j} \right).$$

Finally, Coach Bass needs to add the constraints to the formulation. One set of constraints is almost the same as before: Every leg must have exactly one swimmer assigned to it. The only difference this time is that there are 8 possible swimmers.

However, the other set of constraints is a bit different. It is no longer true that every swimmer must be assigned to exactly one leg because four of the swimmers will not be assigned to any leg. Thus, the second set of constraints is: No swimmer can be assigned to more than one leg.

Because each leg must have exactly one of the 8 swimmers assigned to it, the four leg constraints are:

For j = 1, 2, 3, and 4, $\sum_{i=1}^{8} x_{i,j} = 1$

.

However, because no swimmer can be assigned to more than one event, there are now eight swimmer constraints:

For i = 1, 2, 3, 4, 5, 6, 7, and 8, $\sum_{j=1}^{4} x_{i,j} = 1$

Putting the various parts of the formulation all together, Coach Bass has the following:

Decision Variables

Let

$x_{i,j}$ = the binary decision value representing whether swimmer i is assigned to leg j, for $1 \leq i \leq 8$ and $1 \leq j \leq 4$

z = the total of the best times of the four swimmers assigned to swim the relay

Objective Function

Minimize: $z = \sum_{i=1}^{8} \left(\sum_{j=1}^{4} x_{i,j} \cdot t_{i,j} \right)$, where $t_{i,j}$ = the best time for swimmer i in leg j.

Subject to:

Constraints

for j = 1, 2, 3, and 4, $\sum_{i=1}^{8} x_{i,j} \leq 1$

for i = 1, 2, 3, 4, 5, 6, 7, and 8, $\sum_{j=1}^{4} x_{i,j} = 1$

Figure 6.4.3 shows a spreadsheet formulation of the medley relay team problem with 8 swimmers. Notice that, just as before, the spreadsheet contains two matrices. One matrix has the best times for each event for each swimmer ($t_{i,j}$). Another matrix shows the decision variables, the assignment of swimmers to events ($x_{i,j}$). Each cell in the assignment matrix contains a zero because at the beginning of the solution process, no swimmer has been assigned to any event.

	A	B	C	D	E	F	G	H	I
1	Chapter 6: Binary Programming								
2	6.3.1 Coach Bass's Problem								
3	Relay Time Minimization								
4									
5	**Swimmers' Times ($t_{i,j}$)**								
6			Butterfly (sec)	Backstroke (sec)	Breaststroke (sec)	Freestyle (sec)			
7		Schmidt	59.59	59.83	72.83	52.61			
8		Reid	60.45	59.56	74.14	53.31			
9		Sanchez	61.84	64.63	73.69	53.70			
10		Lamartina	62.37	59.13	74.36	54.77			
11		Wu	60.33	64.3	72.74	54.05			
12		Greene	62.41	59.03	72.19	56.61			
13		Kleinfeld	62.43	67.63	74.05	55.55			
14		Lepinski	59.44	65.06	72.74	52.49		**Total Time (sec)**	
15								0	
16	**Assignments ($x_{i,j}$)**								
17			Butterfly	Backstroke	Breaststroke	Freestyle	**Assigned**		**Capacity**
18		Schmidt					0	≤	1
19		Reid					0	≤	1
20		Sanchez					0	≤	1
21		Lamartina					0	≤	1
22		Wu					0	≤	1
23		Greene					0	≤	1
24		Kleinfeld					0	≤	1
25		Lepinski					0	≤	1
26		**Assigned**	0	0	0	0			
27			=	=	=	=			
28		**Required**	1	1	1	1			

Figure 6.3.3: Spreadsheet formulation of the medley relay team with eight swimmers.

Q16. If Sanchez is assigned to swim the freestyle leg, what value will appear in cell F20?

Q17. What value will appear in every other cell in that row of the assignment matrix?

Q18. What value will appear in every other cell in that column of the assignment matrix?

The objective is to minimize the total of the best times of the four swimmers who will be assigned to the medley relay. The objective function has been stored under the cell labeled "Total Time (sec)."

Q19. Why is there a zero under the cell labeled "Total Time (sec)" in the spreadsheet depicted in Figure 6.3.3?

Notice that the cell directly below each leg column of the assignment matrix contains a 0. The cell below the 0 contains an = sign, and the cell below that contains a 1. Notice also that the row of zeros is labeled "Assigned" and the row of ones is labeled "Required."

Q20. What does that mean in the context of the problem?

Similarly, notice that the cell directly to the right of each row of the assignment matrix contains a 0. The cell to the right of the 0 contains a ≤ sign. Lastly, the cell to the right of that one contains a 1. Also notice that the column of zeros is labeled "Assigned" and the column of ones is labeled "Capacity."

Q21. What does that mean in the context of the problem?

Q22. Why are these eight cells labeled ≤, while the other four were labeled =?

Figure 6.3.4 shows the previous spreadsheet after Excel Solver has found a solution for the problem of assigning eight swimmers to the four legs of the medley relay.

	A	B	C	D	E	F	G	H	I
1	Chapter 6: Binary Programming								
2	6.3.1 Coach Bass's Problem								
3	Relay Time Minimization								
4									
5	**Swimmers' Times ($t_{i,j}$)**								
6			Butterfly (sec)	Backstroke (sec)	Breaststroke (sec)	Freestyle (sec)			
7		Schmidt	59.59	59.83	72.83	52.61			
8		Reid	60.45	59.56	74.14	53.31			
9		Sanchez	61.84	64.63	73.69	53.70			
10		Lamartina	62.37	59.13	74.36	54.77			
11		Wu	60.33	64.3	72.74	54.05			
12		Greene	62.41	59.03	72.19	56.61			
13		Kleinfeld	62.43	67.63	74.05	55.55			
14		Lepinski	59.44	65.06	72.74	52.49		**Total Time (sec)**	
15								243.37	
16	**Assignments ($x_{i,j}$)**								
17			Butterfly	Backstroke	Breaststroke	Freestyle	**Assigned**		**Capacity**
18		Schmidt	0	0	0	1	1	≤	1
19		Reid	0	0	0	0	0	≤	1
20		Sanchez	0	0	0	0	0	≤	1
21		Lamartina	0	1	0	0	1	≤	1
22		Wu	0	0	0	0	0	≤	1
23		Greene	0	0	1	0	1	≤	1
24		Kleinfeld	0	0	0	0	0	≤	1
25		Lepinski	1	0	0	0	1	≤	1
26		**Assigned**	1	1	1	1			
27			=	=	=	=			
28		**Required**	1	1	1	1			

Figure 6.3.4: Optimal assignment of swimmers to the medley relay

Q23. What is the optimal assignment of swimmers to legs of the relay?

Q24. Are there any swimmers in the optimal solution who are not the fastest swimmer for a specific leg? Are they all at least the second fastest swimmer in the event they have been assigned?

Q25. What is the minimum total time?

Q26. What is that time when converted to minutes and seconds?

Section 6.3.2: Ms. Newman Assigns Students to Teams

Ms. Cynthia Newman wants to assign her 20 eighth-grade students to four teams of five students so that they can work on a science project. She wants to be fair about the team assignments, so that no team has an advantage over any other team. She knows that each team must have at least one effective leader to keep the team organized and on-task. She is going to require a written report of the project, so she decides to require at least two students with good writing skills on each team. The project will require some analysis of data that will have to be collected, so she decides to require that each team have two analysts and two data collectors. Finally, she is also going to require each team to make an oral presentation to the rest of the class, so each team is going to need at least two students who are good presenters.

Ms. Newman identifies the roles that she believes each of her students might satisfy based on their skills. For example, she thinks that Don McAllister, one of her students, could play the role of leader, analyst, or data collector.

She identifies at least two possible roles for each of her students. Table 6.3.4 shows the possible roles for each of her students. In the table, Ms. Newman uses a "1" to indicate that she believes that a particular student can fill a particular role. Otherwise, she uses a "0". So, in this sense, these cell values are binary indicators.

		Skill				
		Leader	**Writer**	**Analyst**	**Presenter**	**Data Collector**
Student	**1**	1	0	1	0	1
	2	0	1	0	1	0
	3	0	1	0	0	1
	4	1	0	1	1	0
	5	0	1	0	0	1
	6	0	1	0	1	1
	7	1	0	1	0	1
	8	0	1	0	1	1
	9	0	1	0	0	1
	10	1	0	1	1	0
	11	0	1	0	1	1
	12	0	0	1	0	1
	13	0	1	0	1	0
	14	1	0	1	0	1
	15	0	1	0	1	0
	16	0	0	1	0	1
	17	1	1	0	1	0
	18	0	1	0	1	0
	19	0	1	0	0	1
	20	1	0	1	1	1

Table 6.3.4: Roles Ms. Newman has identified for her students

Q27. Which roles does Ms. Newman believe that student number 17 could fill?

Q28. Which students does Ms. Newman believe could fill the role of "leader"?

Q29. For each role except for "leader", Ms. Newman is going to require that each team have at least two students who can fill that role. Why might it make sense to only require one leader on each team?

Mrs. Newman also does not want to give any team a significant academic advantage. She has the grade point averages (GPAs) for each of her students, but she wonders what the best way to use that information would be. She knows that the average GPA of each of the four teams is certainly going to be different.

She also knows that the average GPA for the entire class is 2.94. Table 6.3.5 contains the GPA for each of her students.

Student	GPA
1	3.13
2	3.43
3	3.70
4	2.84
5	3.81
6	2.70
7	3.75
8	2.35
9	3.44
10	2.42
11	2.04
12	2.46
13	2.47
14	2.39
15	2.92
16	2.24
17	2.14
18	3.82
19	3.20
20	3.64

Table 6.3.5: Class GPAs

Ms. Newman's younger brother, Fred, is a graduate student in operations research. He suggests that she might use a technique called *maximizing the minimum* in order to solve her problem. Fred explains that she could form teams in such a way that the minimum average of each team's GPA would be as large as possible. Of course, the solution would still have to satisfy her other constraints.

Assume, for example, she creates four teams, and. the average GPAs of these teams are 2.75, 2.91, 3.03, and 3.07. The minimum average GPA for these four teams is 2.75. This GPA is substantially lower than the highest average of 3.07. Therefore, this arrangement of teams does not appear to be very fair.

Maximizing the minimum average is one way to establish fairness with respect to the academic records of the four teams. Ms. Newman decides to use Fred's suggestion.

Formulating the Problem

Fred helps his sister formulate her team problem. He recognizes it as an assignment problem with a minimum team GPA objective and five constraints. The decision variables are all binary. A 0 represents the decision not to assign a particular student to a particular team, and a 1 represents assigning that particular student to that team.

Therefore, each student will have four different decision variables, one for each team. When a solution is found, for each student, three out of the four associated decision variables will equal 0; the other will equal 1.

Q30. Suppose student 1 ends up with the decision values found in Table 6.3.6. What does this mean in the context of the problem?

	Team 1	Team 2	Team 3	Team 4
Student 1	0	1	0	0

Table 6.3.6: Example decision values for student 1

Since there are 20 students in Ms. Newman's class and four possible teams, there are a total of $20 \cdot 4 = 80$ decision variables. Now Fred is ready to begin the formulation. First, he defines those decision variables:

For $1 \le i \le 20$ and $1 \le j \le 4$, let $s_{i,j} = \begin{cases} 0, & \text{if student } i \text{ is } \textit{not} \text{ assigned to team } j\text{, and} \\ 1, & \text{if student } i \text{ is assigned to team } j. \end{cases}$

Next, he defines a variable representing the students' GPAs:

For $1 \le i \le 20$, let g_i = the GPA of student i.

Now, since Ms. Newman wants to maximize the minimum GPA for each team, Fred needs to create a formula to compute the average GPA for each team:

For $1 \le j \le 4$, let $A_j = \dfrac{\sum_{i=1}^{20} s_{i,j} \cdot g_i}{5}$.

Recall that the objective here is to have a fair distribution of the students' GPAs as they are assigned to teams. To ensure fairness, Fred has suggested maximizing the minimum average GPA of the four teams. So, the objective function is:

$z = \min (A_1, A_2, A_3, A_4)$.

This notation simply means to find the minimum value of the average GPAs A_1, A_2, A_3, and A_4. Then, the objective is to maximize z, so Fred uses the following notation:

$\max z = \max [\min (A_1, A_2, A_3, A_4)]$,

With the help of Fred, Ms. Newman establishes the basic assignment constraints for students and teams. First, every student must be assigned to exactly one team. So, the first constraint is:

For $1 \le i \le 20$, $\sum_{j=1}^{4} s_{i,j} = 1$.

Next, every team must consist of exactly five students. So, the second constraint is:

For $1 \le j \le 4$, $\sum_{i=1}^{20} s_{i,j} = 5$.

In addition, each team must be made up of the following:

- At least 1 leader
- At least 2 writers
- At least 2 analysts
- At least 2 presenters
- At least 2 data collectors.

In order to formulate these constraints, Fred creates a matrix, $R = r_{i,k}$, using the information in Table 6.3.4. If student i has been identified as capable of handling role k, then the value of $r_{i,k}$ is 1. Otherwise, the value of $r_{i,k}$ is 0. Now for each team, there is a constraint for each possible role.

Q31. How many role constraints are there?
5 constraints: leader, writers, analysts, presenters, and data collectors, for each of the 4 teams is a total of 20

For example, for the role of leader, Ms. Newman must check to see if, among the five students assigned to team 1, there is at least one whom she has identified as a possible leader. Then she must do the same for each of the other four roles. Finally, she must repeat the entire process for each of the other three teams. To formulate all of this, Fred defines the following constraints:
Leader:

For $1 \le j \le 4$, for $k = 1$, $\sum_{i=1}^{20} s_{i,j} \cdot r_{i,1} \ge 1$,

Writer:

For $1 \le j \le 4$, for $k = 2$, $\sum_{i=1}^{20} s_{i,j} \cdot r_{i,2} \ge 2$,

Analyst:

For $1 \le j \le 4$, for $k = 3$, $\sum_{i=1}^{20} s_{i,j} \cdot r_{i,3} \ge 2$,

Presenter:

For $1 \le j \le 4$, for $k = 4$, $\sum_{i=1}^{20} s_{i,j} \cdot r_{i,4} \ge 2$, and

Data Collector:

For $1 \le j \le 4$, for $k = 5$, $\sum_{i=1}^{20} s_{i,j} \cdot r_{i,5} \ge 2$.

Interpreting the Spreadsheet Solution

Figure 6.3.6 shows this formulation in a spreadsheet format.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
1																		
2											TEAMS							
3					ROLES						1	2	3	4				
4					Leader	Writer	Analyst	Presenter	Data Collector	GPA	Decision Variables							
5				1	1		1		1	3.13	0	0	0	1		1	=	1
6				2		1		1		3.43	0	1	0	0		1	=	1
7				3		1			1	3.70	0	1	0	0		1	=	1
8				4	1		1	1		2.84	0	1	0	0		1	=	1
9				5		1			1	3.81	0	0	1	0		1	=	1
10				6		1		1	1	2.70	1	0	0	0		1	=	1
11			S	7	1		1		1	3.75	1	0	0	0		1	=	1
12			T	8		1		1	1	2.35	1	0	0	0		1	=	1
13			U	9		1			1	3.44	1	0	0	0		1	=	1
14			D	10	1		1	1		2.42	0	0	1	0		1	=	1
15			E	11		1		1	1	2.04	0	0	1	0		1	=	1
16			N	12			1		1	2.46	0	0	1	0		1	=	1
17			T	13		1		1		2.47	0	1	0	0		1	=	1
18			S	14	1		1		1	2.39	1	0	0	0		1	=	1
19				15		1		1		2.92	0	0	0	1		1	=	1
20				16			1		1	2.24	0	1	0	0		1	=	1
21				17	1	1		1		2.14	0	0	0	1		1	=	1
22				18		1		1		3.82	0	0	1	0		1	=	1
23				19		1			1	3.20	0	0	0	1		1	=	1
24				20	1		1	1	1	3.64	0	0	0	1		1	=	1
25										2.94	2.93	2.94	2.91	3.01	2.91			
26											A1	A2	A3	A4	min A1:A4			
27					2	3	2	2	5	0.02	5	5	5	5				
28					1	3	2	3	2	0.03	=	=	=	=				
29					1	3	2	3	3	0.00	5	5	5	5				
30					3	3	2	3	3	0.10								
31					>=	>=	>=	>=	>=	>=								
32					1	2	2	2	2	0			2.91					
33													2.91	Optimal				

Figure 6.3.6: Spreadsheet formulation of Ms. Newman's Team Assignment Problem

Q32. Which cells in the spreadsheet contain the values of the decision variables?

Q33. Cell J25 contains the value 2.94. How was that value obtained?

Q34. What do the values in cells K25 through N25 represent?

Q35. In which cell(s) in the spreadsheet is the value of the objective function? How was that cell defined?

Q36. Which cells in the spreadsheet correspond to the constraint that every student must be assigned to exactly one team? How were the left hand sides of those constraints defined?

Q37. What do the values in cells E27 thru I27, E28 thru I28, E29 thru I29, E30 thru I30 represent? How were those values obtained?

Q38. How were the values in cells J27 thru J30 obtained? What do those values tell you?

Q39. The values in the cells of the spreadsheet in Figure 6.4.6 represent an optimal solution. What is that optimal solution? What is the smallest team average GPA that satisfies all of the constraints?

Q40. For the optimal solution, what is the range of the team GPAs. Do you think that this solution meets Ms. Newman's fairness objective? Explain.

A Complication: Students Who Cannot Work Together on a Team

After looking at the spreadsheet solution to her team assignment problem, Ms. Newman realizes that she did not account for some important information. She knows from past experience that students 7 and 8, who were assigned to the same team, cannot work with each other. She also knows that one other pair of students is also incapable of working together: students 17 and 18. She tells her brother Fred about this, to see if there is any way to account for this additional complication. Fred assures her that, although it will add constraints to the formulation, the problem will still be solvable. Figure 6.3.7 contains the spreadsheet after Fred altered it to include the new constraints.

Here is how he handled the "Can't work together" constraints. Since students 7 and 8 cannot work together, Fred does not want to assign them to the same team. To ensure that this does not happen, Fred realized that the sum of cells K11 and K12, L11 and L12, M11 and M12, and N11 and N12 each must be less than or equal to 1. Row 34 contains those sums in columns K through N.

Similarly for students 17 and 18, the sum of cells K21 and K22, L21 and L22, M21 and M22, and N211 and N22 each must be less than or equal to 1. Row 33 contains those sums in columns K through N. Notice that in row 35, columns K through L contain "<=" and row 36, columns K through N are filled with 1s. Thus, rows 33, 35, and 36 contain the restriction that students 17 and 18 cannot be on the same team. Each value in row 33 from columns K thru N must be less than 1 as recorded in row 36. In the same way, rows 34, 35, and 36 contain the restriction that students 7 and 8 cannot be on the same team.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
1																		
2												TEAMS						
3							ROLES				1	2	3	4				
4					Leader	Writer	Analyst	Presenter	Data Collector	GPA	Decision Variables							
5				1	1		1		1	3.13	0	0	0	1		1	=	1
6				2		1		1		3.43	1	0	0	0		1	=	1
7				3		1			1	3.70	0	0	0	1		1	=	1
8				4	1		1	1		2.84	0	0	0	1		1	=	1
9				5		1			1	3.81	0	1	0	0		1	=	1
10				6		1		1	1	2.70	1	0	0	0		1	=	1
11	S	Cannot Work		7	1		1		1	3.75	1	0	0	0		1	=	1
12	T	together		8		1		1	1	2.35	0	0	0	1		1	=	1
13	U			9		1			1	3.44	0	0	1	0		1	=	1
14	D			10	1		1	1		2.42	0	1	0	0		1	=	1
15	E			11		1		1	1	2.04	0	1	0	0		1	=	1
16	N			12			1		1	2.46	0	1	0	0		1	=	1
17	T			13		1		1		2.47	1	0	0	0		1	=	1
18	S			14	1		1		1	2.39	0	0	1	0		1	=	1
19				15		1		1		2.92	0	0	1	0		1	=	1
20				16			1		1	2.24	1	0	0	0		1	=	1
21		Cannot Work		17	1	1		1		2.14	0	0	1	0		1	=	1
22		together		18		1		1		3.82	0	1	0	0		1	=	1
23				19		1			1	3.20	0	0	0	1		1	=	1
24				20	1		1	1	1	3.64	0	0	1	0		1	=	1
25								Average GPA		2.94	2.92	2.91	2.91	3.04	2.91			
26											A1	A2	A3	A4	min A1:A4			
27					1	3	2	3	3	0.01	5	5	5	5				
28	2.91				1	3	2	3	3	0.00	=	=	=	=				
29	2.91	Optimal			3	3	2	3	3	0.00	5	5	5	5				
30					2	3	2	2	4	0.14								
31					>=	>=	>=	>=	>=	>=								
32					1	2	2	2	2	0								
33											0	1	1	0				
34											1	0	0	1				
35											<=	<=	<=	<=				
36											1	1	1	1				

Figure 6.3.7: Spreadsheet Formulation Adjusted to Handle Two New Sets of Constraints

Q41. How many constraints were added? Why so many?

Q42. What does the value 0 mean in cell K33? What does the value 1 mean in cell K34?

Q43. Compare the spreadsheets in Figures 6.3.6 and 6.3.7. What effect(s) did these two new sets of constraints have on the optimal solution? What effect did they have on the resulting Team GPAs?

Section 6.4: Chapter 6 (Binary Programming) Homework Questions

1. The City Council in Monroe, Michigan is considering four proposed new recreational facilities: a swimming pool, a tennis center, athletic fields (football/soccer, baseball/softball), and a gymnasium. The Council wants to construct the facilities that will maximize the expected daily use, but there are budgetary and land restrictions. The expected daily use, cost, and land requirements for each of the proposed facilities are given in Table 1.

Facility	Expected Use (people/day)	Cost ($)	Land Required (acres)
Swimming pool	500	350,000	4
Tennis center	150	100,000	2
Athletic fields	750	250,000	7
Gymnasium	400	500,000	3

Table 6.4.1: Information on proposed recreational facilities

The Council has budgeted $900,000 for the construction of new recreational facilities. There are 12 acres of land available, but only one site that is adequate for the swimming pool or gymnasium. Thus, only one or the other of these two facilities can be built.

a) Define a set of binary decision variables for this problem.
b) Use the decision variables you defined to define the objective function.
c) Formulate the constraints in terms of the decision variables.
d) Enter the problem formulation into a spreadsheet.
e) Use solver to obtain the optimal solution.
f) Which of the proposed facilities should the Council build?

2. The Research Triangle Electronics Company is considering eight new research and development projects. The company cannot conduct all eight projects, due to limitations on their R & D budget and the number of research scientists available. Table 2 contains the resource requirements and estimated profit for each of the projects. In addition, not more than two of projects 4, 5, and 6 can be undertaken, because they require many of the same research scientists. Which projects should be selected in order to maximize estimated profit?

Project	Cost ($1,000s)	Research Scientists Required	Estimated Profit (Millions of $)
1	650	7	8.2
2	1,200	6	9.5
3	350	8	3.7
4	450	9	1.1
5	1,000	10	2.3
6	850	8	2.2
7	750	7	8.2
8	700	4	5.8

Table 6.4.2: Information about eight possible research projects

3. TopTen Recording Studios is considering funding recording projects with four different artists over the next three years. The studio has allocated $12 million per year to cover the expenses of the new projects. The artists, necessary expenditures per year, and the expected profit from their projects are given in Table 3. Which projects should be selected to maximize the expected total profit?

Artist	Expenditures (Millions of $/Year)			Profit (Millions of $)
	Year 1	Year 2	Year 3	
Rambling Lou	3	2	6	32
Rita Rivera	5	5	1	19
Nightrider	6	5	10	33
SoozieQT	3	8	1	23
Available Funds/Year	12	12	12	

Table 6.4.3: Expenditures per year and expected profit from 4 recording projects

a) Define a set of binary decision variables to fit this problem.
b) Use your decision variables to define the objective function.
c) Formulate the annual budget constraints.
d) Use a spreadsheet solver to obtain the optimal solution.
e) Which artists' projects should Top Ten Recording select?

4. **Housework for Kids**. Joanne Mankowski has three sons: Brad, Mike, and Paul. She is getting tired of doing the housework by herself, and she wants her sons' cooperation in keeping the house clean. She offered them payment if they share the housework on the weekend. She determined three types of tasks that are doable for her sons: washing the dishes after dinner, vacuuming the family room, and dusting the furniture in the living room. The kids told Joanne their preferred payment amount for each task secretly. Those amounts are represented in Table 7.4.1. Assign each son to a task so that the assignment generates the least cost to their mom.

($)	Bid

		Dish Washing	Vacuuming	Dusting
Son	Brad	6	11	7
	Mike	9	13	9
	Paul	7	14	10

Table 7.4.1: Preferred payment amount of each son

5. **Flight Attendants**. Triangle Airlines is assigning six new flight attendants to fly on the six types of aircraft flown by the airline. Each of the new attendants has been trained on each type of aircraft, but the number of training hours the new attendants have on the different aircraft varies. The airline wants to assign the attendants based on their number of hours of training on each on each aircraft type. Table 7.4.2 provides the number of training hours on each aircraft type for each attendant. How should the airline assign the attendants if it wants to assign them based on their training experiences? (Hint: Is this a maximization or minimization problem?)

(hours of training)		Aircraft					
		CRJ	DC-9	A320	747	757	767
Attendant	Albert	4	4	2	4	2	8
	Jack	4	4	4	4	4	4
	Mary	4	2	2	4	8	4
	Katie	2	2	4	4	4	8
	Dave	2	2	4	6	6	4
	Matthew	4	2	2	6	6	4

Table 7.4.2: Training experience of the new attendants

6. **Leaders for Projects**. Mr. Summit has four projects, one each in marketing, product development, logistics, and finance. He has chosen four employees with good leadership skills, Ahmad, John, Julia, and Subhash. Now, it is time to assign the right person to the right project. First, he developed a test of 20 questions for each project and asked the four employees to answer all of the questions. He wants to assign the leaders to the tasks so that the total number of mistakes on the test will be the minimum possible. The number of mistakes each employee made on the test is displayed in Table 7.4.3.

(# of mistakes)		Project			
		Marketing	Product Development	Logistics	Finance
Employee	Ahmad	1	2	3	3
	John	4	4	3	4
	Julia	1	2	1	3
	Subhash	4	2	2	2

Table 7.4.3: Number of mistakes made in the test

7. **Industrial Training.** Industrial Training Consultants is offering four types of courses in August and there are five instructors who are specialized in the subjects. The assignment will be done based on past student evaluations of the five instructors. The student evaluation scores appear in Table 7.4.4.

How shall instructors be assigned to courses so that the total of the student evaluation scores is a maximum?

(% positive)		Course			
		Lean Manufacturing	Six Sigma	Logistic Management	Simulation
Student	Randolph	93	96	86	87
	Angela	90	94	92	89
	Anthony	91	87	84	88
	Deborah	92	88	90	85
	Myles	95	97	94	88

Table 7.4.4: Student evaluations

8. **Renovaçión Home Improvement Store**. The Renovaçión Home Improvement Store will assign an employee to each of the five departments: Appliances, Flooring, Outdoor Living, Kitchen, and Tools and Hardware. There are seven employees available who have past experience in all of these five departments. The average daily sales of each employee are shown in Table 7.4.5.
   a. Assign employees to departments so that the average daily sales of the five employees assigned are as great as possible.

($)		Department				
		Appliances	Flooring	Plumbing	Doors	Lighting
Employee	Joshua	1,555	525	370	275	560
	Adan	1,250	450	285	250	540
	Ha	850	500	320	330	550
	Tyson	1,675	490	375	350	580
	Valley	1,125	510	365	345	190
	Lacole	950	500	195	335	350
	Haemon	1,050	300	345	200	545

Table 7.4.5: Average daily sales of employees by department

   b. In the optimal solution, which individuals were the best in their selected category? Which individuals were not the best? Explain why the optimal solution did not pick the best in each.

9. **VogueTech Computer**. The VogueTech Customer Service provides 24-hour online technical support. There are three 8-hour shifts and thirteen representatives. Five representatives will be assigned to the morning shift, five will be assigned to the afternoon shift, and 3 will be assigned to the night shift. The manager wants to assign them according to their preferences. Table 7.4.6 shows the shift preferences of each representative. “5” indicates the most preferred and “1” indicates the least preferred. Assign them according to their preferences.

(rank)		Shift		
		6 a.m. – 2 p.m.	2 p.m. – 10 p.m.	10 p.m. – 6 a.m.
Representative	1	3	5	1
	2	2	4	2

	3	4	1	4
	4	4	4	4
	5	3	1	4
	6	5	5	4
	7	3	4	3
	8	5	5	2
	9	2	1	1
	10	2	2	4
	11	1	4	1
	12	5	4	5
	13	5	1	5

Table 7.4.6: The preferences of each representative

a. Experience shows that representatives 3 and 4 from exercise 6 do not work well together. Therefore, management decides not to assign them to the same shift. How does this affect the optimal solution?

10. **Disaster Kits.** Counselor Cynthia Walker at Foster High School assigns students to community service work as apart of their graduation requirements. She recently received a notice about a project from the Community Help organization to pack kits for disaster relief. A section of the country needs assistance after a hurricane struck the area. Six types of kit are needed: Emergency Food Packs, Children's, Personal Care, Food Support, Layette, and Household. Only one student can be assigned to pack each of the six different types of kit, but ten students have signed up. All of these students have previously packed relief kits for this organization. The organization would like to pack 60 of each type kit.

As a time saving method, the Community Help organization calculates and records the packing rate for each volunteer, in order to assign the most efficient volunteer to the right task. The packing rate is the number of kits packed per hour. The organization would like to assign the volunteers so that 60 of each type kit are packed in the least time.

(minutes)		Kit					
		Emergency Food Packs	Children's	Personal Care	Food Support	Layette	Household
Student	Abdullah	3	6	6	6	4	3
	Susan	12	18	7	9	3	9
	Jeff	14	12	6	10	1	3
	Briana	5	13	6	13	2	4
	Naomi	6	14	9	15	1	9
	Brenden	14	9	2	12	3	9
	Carlos	12	17	11	5	2	1
	LaQuita	4	16	2	14	7	6
	Matthew	4	14	9	6	1	3
	Erika	8	17	4	14	4	3

Table 7.4.7: Packing rates for different kinds of kits

a. How can the Community Help organization determine which student to assign to each task?
b. Which student should be assigned to each task in order to pack the kits in the least amount of time?

11. **School Bus Route Assignment.** The school district in Livonia, Michigan makes annual contracts with school bus companies. There are three companies who are bidding for nine routes in the Livonia School District. First, the companies announce their bids for the routes they are interested in. Then, the city decides which company to assign to each route. The announced bids of each company are shown in Table 7.4.8. A blank cell in the table indicates that the company did not offer a bid for the route. None of the companies that are bidding can be assigned to more than three routes. Help the Livonia Schools assign companies to all routes with a minimum total cost.

($)		Route								
		1	2	3	4	5	6	7	8	9
Company	Never Late	11,000		6,200		8,200	9,850	7,000	9,250	4,900
	Snail's Pace	11,150	8,700		12,950	8,800		6,600	8,400	4,900
	On Time		8,300	6,250	12,600	8,150	9,750	6,500		4,300

Table 7.4.8: Bids on routes of school bus companies

a. Identify the lowest and the second lowest bids on each route.

b. None of the companies can be assigned to more than 3 routes. Assign the companies to the routes manually without violating the 3-route restriction.

c. Formulate the problem.

d. Solve the problem using a spreadsheet solver. Make sure the changing cells include only those routes for which a company actually bid.

12. As you experienced in problem 1 picking the changing cells and writing the equations are tedious. An easier way is to put very high numbers into the blank cells as given in Table 7.4.9.

($)		Route								
		1	2	3	4	5	6	7	8	9
Company	1	11,000	100,000	6,200	100,000	8,200	9,850	7,000	9,250	4,900
	2	11,150	8,700	100,000	12,950	8,800	100,000	6,600	8,400	4,900
	3	100,000	8,300	6,250	12,600	8,150	9,750	6,500	100,000	4,300

Table 7.4.9: Modified bids of school bus companies

a. Why do you think you can find the same solution using the values above?

b. Formulate the problem.

c. Solve it again using Solver. You can use the SUMPRODUCT function to write the objective function.

13. Four more school bus companies are added to the bidding; now there are seven companies that are bidding for nine routes in the Livonia School District. The announced bids are shown in Table 7.4.10. Now each company can be assigned at most two routes. Use the idea from problem 2 to simplify the modeling.

($)		Route								
		1	2	3	4	5	6	7	8	9
Company	1	11,000		5,750		8,200	9,600	5,700	7,600	4,450
	2	10,450	8,600		12,950	8,800		6,600	7,900	4,400
	3		8,350	5,700	12,580	8,150	9,650	5,600		4,600
	4	10,300	8,400	5,675	13,000	8,150	9,750	5,550	7,600	
	5	10,200	8,300	5,600	12,500		9,900		7,575	4,600
	6	10,400	8,500	5,850		8,050	9,500	5,650	7,500	4,300
	7			5,750	12,600	8,750	9,550	5,500	7,700	4,375

Table 7.4.10: Bids of School Bus Companies

a. What will the new assignment be?

b. The city wants to see the affect of assigning each company three routes at most instead of only two. What will the impact be? What if they assign each company four routes at most?

14. Referring to the Homecoming example in section 7.2, what, if anything, would happen to the optimal solution if all of the caterers were able to handle two events? Suppose Campus has been used regularly and has always done an outstanding job. If the committee wanted to assign at least one event to Campus, how, if at all, would doing so change the optimal solution?

Chapter 6 Summary

What have we learned?

This is the last in a series of four chapters on mathematical programming. As in the previous three chapters (LP Max, LP Min, and IP), Binary Programming (BP) is method of modeling a situation in which a decision has to be made to optimize some objective while being constrained by limited resources.

The process for solving a binary integer programming problem is the same as other linear programming problems.

1. Formulate the problem.
    - Identify and define the decision variables.
    - Write the objective function.
    - Identify and write the functional constraints.

2. Enter the problem formulation into a spreadsheet.
    - Enter decision variables, objective function, and constraint coefficients.
    - Create formulas for objective function values and constraints' RHS.
    - Set up Solver Parameters and Options.
    - Add binary constraint for decision variable values.
      Note: Some integer programming problems are maximization and others are minimization problems.
    - Solve and generate Answer Report.

3. Interpret the results.
    - Answer Report shows status and amount of slack for constraints
    - Solver cannot create a Sensitivity Report for binary integer programming problems.

We have also learned that assignment problems are special cases of binary programming problems. The mathematical formulation of these problems has many requirements.

- Matrices are used extensively.
- The binary decision variables are arranged in a compact matrix X. If agent i is assigned to perform task j, then x_{ij} will equal one, otherwise it will equal zero.
- The "cost" of agent i performing task j is element c_{ij} in the cost matrix C.
- The overall "cost" is calculated by taking the sum of all the products of corresponding elements of the decision variable matrix and the cost matrix.

The spreadsheet formulation of assignment problems is somewhat different than in previous chapters.

- The SUMPRODUCT formula needs to be used for the objective function.
- Constraints typically involve row and column totals from the decision variable matrix.

Terms

Assignment Problem An assignment problem arises whenever a number of agents must be paired with a number of tasks

Agent In an assignment problem, the agents are the ones able to perform the tasks

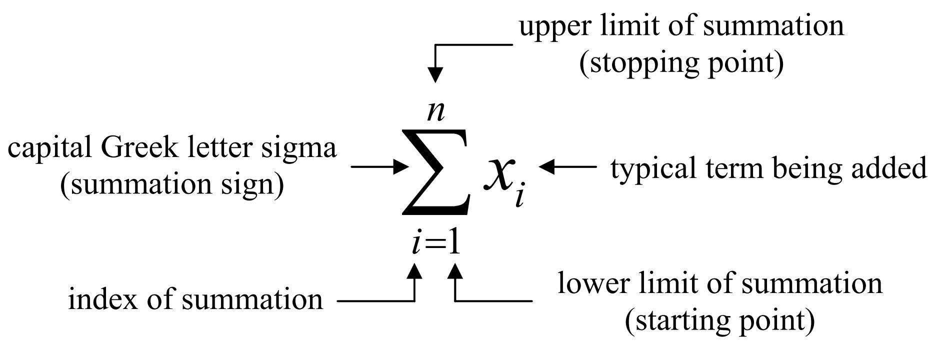
Binary Decision Variable A decision variable that can take on only two possible values, zero or one.

Binary Indicator Coefficient A coefficient that takes the value of one if the quantity meets a given condition or zero if it does not.

Cost The "cost" depends on the context and units of the problem, but it represents the amount of the objective quantity required for an agent to perform a task

Cost Matrix For an assignment problem with *m* agents and *n* tasks, the cost matrix *C* will be an $m \times n$ matrix, and element c_{ij} will represent the cost of agent *i* performing task *j*

Summation Notation



$$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \ldots + x_{n-1} + x_n$$

Task In an assignment problem, the tasks are those things needing to be accomplished

Notation

If $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2n} \\ c_{31} & c_{32} & c_{33} & \dots & c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & c_{m3} & \dots & c_{mn} \end{bmatrix}$ and $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \dots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & x_{m3} & \dots & x_{mn} \end{bmatrix}$, then

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{n} c_{ij} \cdot x_{ij}\right) = \begin{bmatrix} c_{11} \cdot x_{11} & c_{12} \cdot x_{12} & c_{13} \cdot x_{13} & \dots & c_{1n} \cdot x_{1n} \\ c_{21} \cdot x_{21} & c_{22} \cdot x_{22} & c_{23} \cdot x_{23} & \dots & c_{2n} \cdot x_{2n} \\ c_{31} \cdot x_{31} & c_{32} \cdot x_{32} & c_{33} \cdot x_{33} & \dots & c_{3n} \cdot x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{m1} \cdot x_{m1} & c_{m2} \cdot x_{m2} & c_{m3} \cdot x_{m3} & \dots & c_{mn} \cdot x_{mn} \end{bmatrix} = \text{SUMPRODUCT}(C, X).$$

Chapter 6 (Binary Programming) Objectives

You should be able to:

- Identify the objective of the problem
- Identify and define the binary decision variables
- Write the objective function, including using summation notation and double summation notation
- Identify the constraints involved in the problem
    - Standard assignment constraints
        - row totals ≤ 1 indicating each agent can be assigned to at most one task
        - column totals = 1 indicating that each task must be completed by exactly one agent
    - Extra constraints (e.g., modifications to totals, binary indicator coefficients)
- Write the functional constraints as inequalities, including using summation notation
- Use binary indicator coefficients in writing constraints
- Formulate the problem using a compact matrix
    - Each row represents an agent
    - Each column represents a task
- Enter the problem formulation into Excel
- Use SUMPRODUCT formula in Excel
- Set up Solver Parameters and Options, including the constraints that decision variables are binary
- Interpret the optimal solution in the context of the problem
- Analyze the Answer Report

Chapter 6 Study Guide

1. What is the objective function?

2. What are decision variables?

3. What is different about the decision variables in a binary programming (BP) problem compared to an integer programming (IP) problem?

4. What are functional constraints?

5. Besides functional constraints, describe two other types of constraints found in BP problems.

6. Consider a BP problem whose decision variables are of the form x_i, and one of the functional constraints is $\sum_{i=1}^{12} x_i \leq 7$.
   a. Explain how to determine the number of decision variables in the problem?

   b. Interpret the meaning of the constraint.

7. What information is found in the Answer Report for a BP problem?

8. In a BP problem, is it possible for a binary constraint to be nonbinding? Explain why or why not.

9. To what can the "Final Value" on the Answer Report refer?

10. Define *slack*.

11. In the Answer Report, what information is in the "Cell Value" column?

12. Explain how Excel would execute the command "=SUMPRODUCT(A1:B3,D1:E3)."

13. If an assignment problem has nine agents to assign to seven tasks, why is it convenient to use matrices to formulate the problem?

14. Assuming that each agent can perform at most one task and each task must be performed, what is the relationship between the number of agents and the number of tasks in an assignment problem? Why?

15. If an assignment problem has six agents, four tasks, decision variable matrix X, and cost matrix A, use double summation notation to write the objective function.

16. Consider an assignment problem with more agents than tasks. Explain why it is typical that the constraints based on row (agent) totals are inequalities while the constraints based on column (task) totals are equations.

17. Explain the "maximize the minimum" technique that was used in Problem 6.3 (assigning students to teams).

18. Do all assignment problems have a unique solution? Explain why or why not.

References

Summation Notation. (n.d.). . Retrieved July 11, 2011, from http://www.columbia.edu/itc/sipa/math/summation.html

Appendix A: Summation Notation

Summation notation is often used as a simpler and more convenient way to write out a long sum in which all of the terms in the sum have some common feature. For example, in Section 6.1: Flipping Houses, the sum

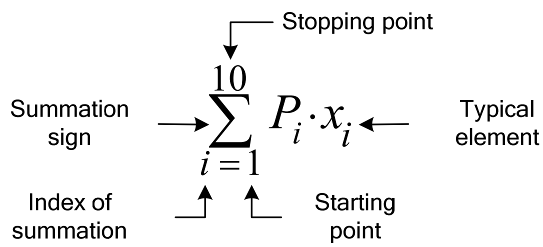
$$z = P_1 \cdot x_1 + P_2 \cdot x_2 + P_3 \cdot x_3 + P_4 \cdot x_4 + P_5 \cdot x_5 + P_6 \cdot x_6 + P_7 \cdot x_7 + P_8 \cdot x_8 + P_9 \cdot x_9 + P_{10} \cdot x_{10}$$

could instead be written as:

$$z = \sum_{i=1}^{10} P_i \cdot x_i$$

In this sum, the common feature is: a product. The factors that are multiplied are an estimated profit (P_i) and a binary decision variable (x_i).

Summation notation, such as this, involves five main elements, as shown below:



The *summation sign* is the Greek letter ∑ (Sigma). To the right of the summation sign is a *typical element* of the sequence of terms being summed. In the Flipping Houses example, the typical element being summed is the estimated profit for a particular house i (P_i) multiplied by the binary decision value for that particular house i (x_i). That pattern can be seen when all of the terms in the sum are completely written out.

The index i plays two roles. First, it specifies a particular term in the sum. In the Flipping Houses example, a particular term can be thought of as a particular house. The second role of the index is to determine how many terms are in the sum. In the above example, this index is i. The *index of summation* is found below the summation sign, along with the starting point. The *starting point* (sometimes called the "lower limit of summation") refers to the first value that i takes on. Then, the *stopping point*, found above the summation sign, refers to the last value that i takes on. In the Flipping Houses example, the sum begins with house 1 (i = 1), ends with house 10 (i = 10), and therefore includes 10 terms.

Consider the following examples.

- The sum $\sum_{i=1}^{5} x_i$ can be rewritten as $\sum_{i=1}^{5} x_i = x_1 + x_2 + x_3 + x_4 + x_5$.
- The sum $\sum_{i=4}^{7} x_i$ can be rewritten as $\sum_{i=4}^{7} x_i = x_4 + x_5 + x_6 + x_7$.
- The sum $\sum_{i=1}^{5} x_i^2$ can be rewritten as $\sum_{i=1}^{5} x_i^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$
- The sum $x_1 + x_2 + x_3 + x_4$ can be rewritten as $\sum_{i=1}^{4} x_i$.
- The sum $x_3 + x_4 + x_5$ can be rewritten as $\sum_{i=3}^{5} x_i$.